NAKED SINGULARITY AND BLACK HOLE IN SELF-SIMILAR SPACETIMES

PhD DISSERTATION

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DEPARTMENT OF PHYSICS
UNIVERSITY OF YANGON
MYANMAR

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ACKNOWLEDGEMENTS

I wish to express my sincere thanks to Dr Win Win Thar,PhD *Ygn*, Professor and Head of Department in Physics, *Ygn*, for her kind permission to present this research paper.

I am also indebted to my supervisor Dr Thant Zin Naing PhD *Ygn*, Professor, Department of Physics, Dawei University, for his close guidance, helpful advice.

I am also indebted to Visiting Professor Dr I H Dwivedi (TIFR Bombay, India) for his guidance and helpful advice.

ABSTRACT

Attempts are made to give a clear description of self-similar spacetimes, which is proving to be very useful in astrophysics and general relativity. We have attempted to study the nature of naked singularity and black hole, which are also likely to take place in self-similar spacetimes. The metric for collapsing dust cloud is utilized in this formalism. Mathematica was used to compute the details of the formalism and to visualize the graphics.

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Gravitational Collapse in Self-Similar Spacetimes

1.1 Introduction

Gravitational collapse is an important issue in general relativity and it is widely believed that it may be responsible for high energy objects in our universe. Energy theorems in relativity have shown that under reasonable energy conditions a matter cloud with sufficient mass would undergo a gravitational collapse. General relativistic field equations involve a system of highly nonlinear partial differential equations and hence analyzing a gravitational collapse scenario in general even in spherically symmetric spacetime is virtually impossible. Self-similar spacetimes have therefore been given considerable attention in recent applications. Due to the symmetry property of

self-similarity equations in self-similar spacetime become an ordinary differential equation and therefore the study of a phenomena becomes much easier to analyze. In this study we therefore use self-similar spherically symmetric spacetimes to examine the gravitational collapse and its features.
In astrophysics and cosmology the self-similar models are of great interest to the relativists and cosmologist alike. It is worthy to start with the very definition of self-similar spacetimes. A self-similar spacetimes is characterized by the existence of a homothetic Killing vector field (Joshi, 1993).

1.2 Self-Similar Spacetimes and Path of Photon

A spherical symmetric spacetime is self-similar if it admits a radial area coordinate r and an orthogonal time coordinate t such that for the metric components g_{tt} and g_{rr} we have

$$g_{tt}(kt, kr) = g_{tt}(t, r)$$

$$g_{rr}(kt, kr) = g_{rr}(t, r)$$

for all $k > 0$. Thus, along the integral curves of the Killing vector field all points are similar.
A spherical symmetric spacetime (SSS) in co-moving coordinates is given

by general form

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2C(t,r)d\Omega^2$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. If SSS is self-similar, self-similarity condition must hold, it must have a homothetic Killing vectors, which means $t \Rightarrow kt$, $r \Rightarrow kr$ and the metric becomes,

$$ds^2 = -A(kt,kr)dt^2 + B(kt,kr)dr^2 + r^2C(kt,kr)d\Omega^2$$

and parameters A,B,C are such that

$$A(t,r) = A(kt,kr)$$

$$B(t,r) = B(kt,kr)$$

$$C(t,r) = C(kt,kr).$$

The collapsing dust cloud is described by the self-similar metric,

$$ds^2 = -dT^2 + R'^2 dr^2 + R^2(d\theta^2 + sin^2\theta d\phi^2) \qquad (1.1)$$

where

$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{F}(t_0(r) - t)$$
$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{F}(a - \frac{t}{r})r^{\frac{3}{2}}$$
$$R = (\frac{3}{2})^{\frac{2}{3}}F(a - \frac{t}{r})^{\frac{2}{3}}r$$

To check the metric, one can proceed as follows:

$$A(t,r) = -1$$

$$A(kt,kr) = -1$$

$$A(t,r) = A(kt,kr)$$

$$B(t,r) = R'^2 = (\frac{3}{2})^{\frac{2}{3}} F(a - \frac{t}{r})^{-\frac{1}{3}}((\frac{2t}{3r}) + (a - \frac{t}{r}))$$

$$B(kt,kr) = (\frac{3}{2})^{\frac{2}{3}} F(a - \frac{kt}{kr})^{-\frac{1}{3}}((\frac{2kt}{3kr}) + (a - \frac{kt}{kr})) = R'^2$$

$$B(t,r) = B(kt,kr)$$

$$C(t,r) = \frac{R^2}{r^2} = (\frac{3}{2})^{\frac{4}{3}} F^2(a - \frac{t}{r})^{\frac{4}{3}}$$

$$C(kt,kr) = (\frac{3}{2})^{\frac{4}{3}} F^2(a - \frac{kt}{kr})^{\frac{4}{3}} = \frac{R^2}{r^2}$$

$$C(t,r) = C(kt,kr)$$

The above shows that the given metric is self-similar. The radial null geodesics in this metric is defined by $ds^2 = 0$ and $k^\theta = k^\phi = 0$ (Tolman, 1934). The geodesic equations for k^T and k^r from Lagrangian equation are

$$L = -k^{T^2} + R'^2 k^{r^2} + R^2 k^{\theta^2} + R^2 sin^2\theta k^{\phi^2} \tag{1.2}$$

$$\frac{\partial L}{\partial T} = \frac{d}{d\lambda}\{\frac{\partial L}{\partial k^T}\}$$

$$k^{r^2}\frac{\partial R'^2}{\partial T} = \frac{d}{d\lambda}\{\frac{\partial(-k^{T^2})}{\partial k^T}\}$$

$$\frac{dk^T}{d\lambda} + R'\dot{R}'k^{r^2} = 0 \tag{1.3}$$

Similarly,

$$\frac{\partial L}{\partial r} = \frac{d}{d\lambda}\{\frac{\partial L}{\partial k^r}\}$$

$$k^{r^2}\frac{\partial R'^2}{\partial r} = \frac{d}{d\lambda}\{\frac{\partial(R'^2 k^{r^2})}{\partial k^r}\}$$

$$2R'R''k^{r^2} = 4R'\frac{dR'}{d\lambda}k^r + 2R'^2\frac{dk^r}{d\lambda}$$

$$\frac{dk^r}{d\lambda} - \frac{R''}{R'}k^{r^2} + \frac{2}{R'}k^r(\dot{R}'k^T + R''k^r) = 0$$

$$\frac{dk^r}{d\lambda} + \frac{2\dot{R}'}{R'}k^T k^r + \frac{R''}{R'}k^{r^2} = 0 \tag{1.4}$$

Let k^a be tangent to radial null geodesics (i.e,$k^a k_a = 0 = k^a_{;b}k^b$) for the metric in eqn (1.1) and $g_{ab}k^a k^b = 0$, for null condition. we have for radial null geodesics from eqns (1.3) and (1.4),

$$g_{TT}k^{T^2} + g_{rr}k^{r^2} = 0$$

$$g_{TT}k^{T^2} = -g_{rr}k^{r^2}$$

$$\frac{k^{T^2}}{k^{r^2}} = -\frac{g_{rr}}{g_{TT}}$$

$$\frac{k^T}{k^r} = \sqrt{\frac{-g_{rr}}{g_{TT}}} = R'$$

$$k^r = \frac{1}{R'}k^T$$

$$\frac{k^T}{k^r} = R' \Rightarrow \frac{dT}{dr} = R'$$

$$\frac{dk^T}{d\lambda} = -R'\dot{R}'k^{r^2}$$

where λ is affine parameter. The Kretschmann scalar of the metric is obtained as

$$K = \frac{16(21a^2r^2 - 10art + 5t^2)}{27(-3ar + t)^2(-ar + t)^4}$$

If we assume that $r = r$ and $t = ar$,we get

$$K = \frac{16(21a^2r^2 - 10a^2r^2 + 5(a^2r^2)}{27(-3ar + ar)^2(-ar + ar)^4}$$

which gives us,

$$K = \infty$$

Therefore,the points of unboundedness i.e., $K = \infty$ occur at (ar, r). Self-similarly implies that all variables of physical interest may be expressed in terms of the similarity parameter $X = \frac{t}{r}$.

$$X = \frac{t}{r}, t = Xr$$

$$dt = Xdr + rdX$$

$$\frac{dt}{dr} = R'$$

$$\frac{Xdr}{dr} + \frac{rdX}{dr} = R'$$

$$X + \frac{rdX}{dr} = R'$$

$$\frac{rdX}{dr} = R' - X$$

$$\frac{dX}{dr} = \frac{R' - X}{r}$$

$$\int \frac{1}{r} dr = \int \frac{1}{R' - X} dX$$

$$\ln r = -\ln[-R' + X]$$

Using the original values of R' and X, one finally obtains,

$$r = -3\ 2^{\frac{1}{3}}\ 3^{\frac{2}{3}}\ aF\ r + 2^{\frac{1}{3}}\ 3^{\frac{2}{3}}\ F\ t + 6\ r(a - \frac{t}{r})^{\frac{1}{3}} x$$

The above is the possible path of photon in gravitationally collapsing objects.

1.3 Gravitational Collapse in Self-Similar Space-times

We would like to examine the determination of curvature strength of the naked singularity in order to decide on its seriousness and physical relevance and the mathematical calculations of Christoffel Symbols, Riemann Tensors, Ricci Tensors and Kretschmann Scalar of the metric.

The collapsing dust cloud is described by the self-similar metric,

$$ds^2 = -dT^2 + R'^2 dr^2 + R^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.5}$$

where

$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{F}(t_0(r) - t)$$
$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{F}(a - \frac{t}{r})r^{\frac{3}{2}}$$
$$R = (\frac{3}{2})^{\frac{2}{3}}F(a - \frac{t}{r})^{\frac{2}{3}}r$$

Let $b = (\frac{3}{2})^{\frac{2}{3}}F$ and which gives

$$R = b(a - \frac{t}{r})^{\frac{2}{3}}r$$

Here,b is constant and after differentiation we get

$$R' = \frac{b(3ar - t)}{3r(a - \frac{t}{r})^{\frac{2}{3}}} \tag{1.6}$$

From metric,

$$g_{11} = -1, \quad g_{22} = R'^2$$

$$g_{33} = R^2, \qquad g_{44} = R^2 sin^2\theta$$

To find some calculations, we have to use the following equations,

Christoffel Symbol are denoted by,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda k} \left(g_{k\nu,\mu} + g_{\mu k,\nu} - g_{\mu\nu,k} \right)$$

Riemann Tensor

$$R_{abcd} = g_{d\epsilon} R^{\epsilon}_{abc}$$

Kritchmann Scalar

$$K = R^{abcd} R_{abcd}$$

Ricci Tensor

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}$$

It is necessary to compute the Christoffel symbols for equation (1.1), from which we can get the curvature tensor. If we use labels (1,2,3,4) for (T, r, θ, ϕ) in the usual way, non zero Christoffel symbols are given by using Tensorpak.m package, namely,

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}, g^{11}(g_{11,1} + g_{11,1} - g(11,1)) + \frac{1}{2} g^{12}(g_{21,1} + g_{12,1} - g_{11,2}), \\
\Gamma^2_{12} &= \frac{2t}{3(3a^2r^2 - 4art + t^2)}, \\
\Gamma^3_{13} &= \frac{2}{-3ar + 3t},
\end{aligned}$$

$$
\begin{aligned}
\Gamma^4_{14} &= \frac{2}{-3ar+3t}, \\
\Gamma^2_{21} &= \frac{2t}{3(3a^2r^2-4art+t^2)}, \\
\Gamma^1_{22} &= \frac{2b^2(3ar-t)t}{27r^2(ar-t)(a-\frac{t}{r}^{\frac{2}{3}})}, \\
\Gamma^2_{22} &= \frac{2t^2}{-9a^2r^3+12ar^2t-3rt^2}, \\
\Gamma^3_{23} &= \frac{3ar-t}{3ar^2-3rt}, \\
\Gamma^4_{24} &= \frac{3ar-t}{3ar^2-3rt}, \\
\Gamma^3_{31} &= \frac{2}{-3ar+3t}, \\
\Gamma^3_{32} &= \frac{3ar-t}{3ar^2-3rt}, \\
\Gamma^1_{33} &= -\frac{2}{3}b^2r(a-\frac{t}{r})^{\frac{1}{3}}, \\
\Gamma^2_{33} &= \frac{3r(-ar+t)}{3ar-t}, \\
\Gamma^4_{34} &= \cot\theta, \\
\Gamma^4_{41} &= \frac{2}{-3ar+3t}, \\
\Gamma^4_{42} &= \frac{3ar-t}{3ar^2-3rt}, \\
\Gamma^4_{43} &= \cot\theta, \\
\Gamma^1_{44} &= -\frac{2}{3}b^2r(a-\frac{t}{r})^{\frac{1}{3}}\sin^2\theta, \\
\Gamma^2_{44} &= -\frac{3r(ar-t)\sin^2\theta}{3ar-t}, \\
\Gamma^3_{44} &= -\cos\theta\sin\theta,
\end{aligned}
$$

From these we get the following nonvanishing components of the Riemann tensor

$$
\begin{aligned}
R_{1212} &= \frac{2(3ar+t)}{9(3ar-t)(-ar+t)^2}, \\
R_{1221} &= -\frac{2(3ar+t)}{9(3ar-t)(-ar+t)^2}, \\
R_{1313} &= -\frac{2}{9(-ar+t)^2}, \\
R_{1331} &= \frac{2}{9(-ar+t)^2}, \\
R_{1414} &= -\frac{2}{9(-ar+t)^2}, \\
R_{1441} &= \frac{2}{9(-ar+t)^2}, \\
R_{2112} &= -\frac{2(3ar+t)}{9(3ar-t)(-ar+t)^2}, \\
R_{2121} &= \frac{2(3ar+t)}{9(3ar-t)(-ar+t)^2}, \\
R_{2323} &= \frac{4t}{9(-3ar+t)(-ar+t)^2}, \\
R_{2332} &= -\frac{4t}{9(-3ar+t)(-ar+t)^2}, \\
R_{2424} &= \frac{4t}{9(-3ar+t)(-ar+t)^2}, \\
R_{2442} &= -\frac{4t}{9(-3ar+t)(-ar+t)^2}, \\
R_{3113} &= \frac{2}{(3ar-3t)^2}, \\
R_{3131} &= -\frac{2}{(3ar-3t)^2}, \\
R_{3223} &= -\frac{4t}{9(-3ar+t)(-ar+t)^2}, \\
R_{3232} &= \frac{4t}{9(-3ar+t)(-ar+t)^2},
\end{aligned}
$$

$$
\begin{aligned}
R_{3434} &= \frac{4}{9(-ar+t)^2},\\
R_{3443} &= -\frac{4}{9(-ar+t)^2},\\
R_{4114} &= \frac{2}{(3ar-3t)^2},\\
R_{4141} &= -\frac{2}{(3ar-3t)^2},\\
R_{4224} &= -\frac{4t}{9(-3ar+t)(-ar+t)^2},\\
R_{4242} &= \frac{4t}{9(-3ar+t)(-ar+t)^2},\\
R_{4334} &= -\frac{4}{9(-ar+t)^2},\\
R_{4343} &= \frac{4}{9(-ar+t)^2},
\end{aligned}
$$

We get non-zero values of Ricci Tensor

$$R_{11} = -\frac{2}{3(3a^2r^2 - 4art + t^2)} \tag{1.7}$$

$$R_{22} = \frac{2b^2(-3ar+t)}{27r^2(ar-t)(a-\frac{t}{r}^{\frac{2}{3}})} \tag{1.8}$$

$$R_{33} = -\frac{2b^2r(a-\frac{t}{r})^{\frac{1}{3}}}{9ar-3t} \tag{1.9}$$

$$R_{44} = -\frac{2b^2r(a-\frac{t}{r})^{\frac{1}{3}}sin^2\theta}{9ar-3t} \tag{1.10}$$

Scalar Curvature is,

$$-\frac{4}{3(3a^2r^2 - 4art + t^2)}$$

and Kritchman Scalar is,

$$\frac{16(21a^2r^2 - 10art + 5t^2)}{27(-3ar+t)^2(-ar+t)^4}$$

We have the geodesics equations for k^T and k^r from Lagrangian equations:

$$\frac{dk^T}{d\lambda} = -R'\dot{R}'k^{r^2} \tag{1.11}$$

For radial null geodesics, one has,

$$\frac{k^T}{k^r} = R' \Rightarrow k^r = \frac{1}{R'}k^T \tag{1.12}$$

Solving the above equations we get

$$k^T = \left(\int \frac{\dot{R}'}{R'}d\lambda\right)^{-1} \tag{1.13}$$

$$k^r = \frac{1}{R'}\left(\int \frac{\dot{R}'}{R'}d\lambda\right)^{-1} \tag{1.14}$$

$$\frac{dT}{dr} = R' = \frac{1}{C(a-X)^{\frac{1}{3}}} = f(X) \tag{1.15}$$

The quantity C here is defined by $C = \frac{3r}{b(3ar-t)}$.

From the equation for dt/dr, it is positive sign solutions, which are outgoing and terminate at the singularity with positive value of X.

The point $t = 0, r = 0$ is a singular point of the above differential equations. The nature of the limiting value of similarity parameter $X = \frac{t}{r}$ plays an important role in the analysis of non-spacelike curves that terminate at the singularity and reveals the exact nature of the singularity. Using equation and l'Hospital's rule we get

$$X_0 = \lim_{t\to 0, r\to 0} \frac{t}{r} = \lim_{t\to 0, r\to 0} \frac{dt}{dr} = \frac{1}{C(a-X_0)^{\frac{1}{3}}}$$

$$\Rightarrow f(X_0) = 0 \tag{1.16}$$

If $f(X) = 0$ has a real root X_0 which give the direction tangent to the integral curves at the singularity. It is possible that a single null geodesics ($X = X_0$) in the (t, r) plane would escape from the singularity.

We shall restrict to positive sign solutions which represent outgoing geodesics, the equation of outgoing geodesics in the for $r = r(X)$ from the above $X = \frac{t}{r}$, we can write using eqns (1.8) and (1.11)

$$\frac{dX}{dr} = \frac{1}{r}\left[\frac{1}{C(a-X)^{\frac{1}{3}}} - X\right] \tag{1.17}$$

Integration of the above equation yields the equation of geodesics, which can be written as

$$r = Dexp\left(\int \frac{C(a-X)^{\frac{1}{3}}}{1 - XC(a-X)^{\frac{1}{3}}} dX\right) \tag{1.18}$$

Here D is a constant that labels different integral curves. We have already established the fact that if the singularity is to be naked, $f(X) = 0$ must have at least one real positive root X_0.

$$r = Dexp\left[\frac{3(a-X)^{\frac{1}{3}}}{XC^3} + \frac{3(a-X)^{\frac{2}{3}}}{2XC^2} + \frac{a-X}{XC} + \frac{3log[-1+(a-X)^{\frac{1}{3}}XC]}{XC^4}\right] \tag{1.19}$$

We now examine the curvature strength of the naked singularity forming at $t = 0, r = 0$.Because even though a naked singularity may form during gravitational collapse, if it is not a strong curvature singularity, it may not be considered a serious counter-example to the cosmic censorship hypothesis. A sufficient condition to ensure a strong curvature singularity is given as

$$\lim_{k\to 0} k^2 R_{ab} k^a k^b \neq 0$$

along at least one non-spacelike geodesic terminating at the singularity with the value of the affine parameter $k = 0$ at the singularity. The stronger sense in which a strong curvature singularity is defined is given by the requirement that the above limiting condition must be satisfied along all non-spacelike geodesics terminating at the naked singularity in the past(Tipler, 1977). Such a strength was examined earlier for the self-similar case along the radial null geodesic and also along two other simple null geodesics (Ori and Piran, 1990) that either come out from or fall into the singularity at (0,0).
For radial null geodesic, the metric is defined by $ds^2 = 0$ and $k^\theta = k^\phi = 0$ (Tolman, 1934). The curvature strength for the self-similar case along the radial null geodesic of spherical symmetric spacetimes is

$$\psi = R_{ab}k^a k^b = R_{11}k^{T^2} + R_{22}k^{r^2} \tag{1.20}$$

where R_{ab} is Ricci tensor. Substitute eqns (1.7)(1.8)and (1.10) in eqn (1.20), we get

$$\psi = -\frac{2}{3(3a^2r^2 - 4art + t^2)}R'^2 k^{r^2} + \frac{2b^2(-3ar + t)}{27r^2(ar - t)(a - \frac{t}{r})^{(\frac{2}{3})}}k^{r^2} \tag{1.21}$$

and we used eqn (1.2) which gives

$$\psi = \frac{-2b^2(3ar - t) + 2b^2(t - 3ar)}{27r^2(3ar - t)(a - X)^{\frac{2}{3}}}$$

$$\psi = \left[\frac{4b^2(-3ar+t)}{27r^2(ar-t)(a-X)^{\frac{2}{3}}} \right] k^{r^2} \tag{1.22}$$

The $\lim_{k \to 0} k^2 \psi$ in general can be computed as follow. Multiplying ψ by k^2 and taking the limit as $k \to 0$, the above gives

$$\lim_{k \to 0} k^2 R_{ab} k^a k^b = \lim_{k \to 0} k^2 \psi$$

$$= \lim_{k \to 0} \left[\frac{4b^2(-3ar+t) k^{r^2} k^2}{27r^2(ar-t)(a-X)^{\frac{2}{3}}} \right] \tag{1.23}$$

Using the fact that as singularity is approached, $k \to 0, r \to 0$ and $X \to X_0$ (positive root) and using l'Hospital's rule, we observe

$$\lim_{k \to 0} k^2 \psi = \lim_{k \to 0} \left[\frac{4b^2(-3ar+t) k^{r^2} k^2}{27r^2(ar-t)(a-X_0)^{\frac{2}{3}}} \right] > 0 \tag{1.24}$$

Thus along radial null geodesics strong curvature condition is satisfied. The above condition along future directed radial null geodesics coming out from the singularity.

```
c = {t,r,θ,ϕ}
```

{t, r, θ, ϕ}

```
gmat = {{-1,0,0,0},{0,S^2,0,0},{0,0,R^2,0},{0,0,0,R^2Sin[θ]^2}}
```

$\{\{-1, 0, 0, 0\}, \{0, S^2, 0, 0\}, \{0, 0, R^2, 0\}, \{0, 0, 0, R^2 \mathrm{Sin}[\theta]^2\}\}$

```
g = {-1,0,0,0,0,S^2,0,0,0,0,R^2,0,0,0,0,R^2Sin[θ]^2}
```

$\{-1, 0, 0, 0, 0, S^2, 0, 0, 0, 0, R^2, 0, 0, 0, 0, R^2 \mathrm{Sin}[\theta]^2\}$

```
<<tensorpak.m
```

```
gupper[c, g]
```

$\left\{\{-1, 0, 0, 0\}, \left\{0, \frac{1}{S^2}, 0, 0\right\}, \left\{0, 0, \frac{1}{R^2}, 0\right\}, \left\{0, 0, 0, \frac{\mathrm{Csc}[\theta]^2}{R^2}\right\}\right\}$

```
R = b*r*(a-t/r)^(2/3)
```

$b\, r \left(a - \frac{t}{r}\right)^{2/3}$

```
S = Simplify[D[R, r]]
```

$\frac{b\,(3\,a\,r - t)}{3\,r\,\left(a - \frac{t}{r}\right)^{1/3}}$

ChristoffelSymbol2[c, g]

$\{12,\ 2\} = \dfrac{2\,t}{3\,(3\,a^2\,r^2 - 4\,a\,r\,t + t^2)}$

$\{13,\ 3\} = \dfrac{2}{-3\,a\,r + 3\,t}$

$\{14,\ 4\} = \dfrac{2}{-3\,a\,r + 3\,t}$

$\{21,\ 2\} = \dfrac{2\,t}{3\,(3\,a^2\,r^2 - 4\,a\,r\,t + t^2)}$

$\{22,\ 1\} = \dfrac{2\,b^2\,(3\,a\,r - t)\,t}{27\,r^2\,(a\,r - t)\,(a - \frac{t}{r})^{2/3}}$

$\{22,\ 2\} = \dfrac{2\,t^2}{-9\,a^2\,r^3 + 12\,a\,r^2\,t - 3\,r\,t^2}$

$\{23,\ 3\} = \dfrac{3\,a\,r - t}{3\,a\,r^2 - 3\,r\,t}$

$\{24,\ 4\} = \dfrac{3\,a\,r - t}{3\,a\,r^2 - 3\,r\,t}$

$\{31,\ 3\} = \dfrac{2}{-3\,a\,r + 3\,t}$

$\{32,\ 3\} = \dfrac{3\,a\,r - t}{3\,a\,r^2 - 3\,r\,t}$

$\{33,\ 1\} = -\dfrac{2}{3}\,b^2\,r\,\left(a - \dfrac{t}{r}\right)^{1/3}$

$\{33,\ 2\} = \dfrac{3\,r\,(-a\,r + t)}{3\,a\,r - t}$

$\{34,\ 4\} = \mathrm{Cot}[\theta]$

$\{41,\ 4\} = \dfrac{2}{-3\,a\,r + 3\,t}$

$\{42,\ 4\} = \dfrac{3\,a\,r - t}{3\,a\,r^2 - 3\,r\,t}$

$\{43,\ 4\} = \mathrm{Cot}[\theta]$

$\{44,\ 1\} = -\dfrac{2}{3}\,b^2\,r\,\left(a - \dfrac{t}{r}\right)^{1/3}\,\mathrm{Sin}[\theta]^2$

$\{44,\ 2\} = -\dfrac{3\,r\,(a\,r - t)\,\mathrm{Sin}[\theta]^2}{3\,a\,r - t}$

$\{44,\ 3\} = -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

Riemann[c, g]

$\text{Riemann }[12,12] = \dfrac{2\,(3\,a\,r + t)}{9\,(3\,a\,r - t)\,(-a\,r + t)^2}$

$\text{Riemann }[12,21] = -\dfrac{2\,(3\,a\,r + t)}{9\,(3\,a\,r - t)\,(-a\,r + t)^2}$

$\text{Riemann }[13,13] = -\dfrac{2}{9\,(-a\,r + t)^2}$

$\text{Riemann }[13,31] = \dfrac{2}{9\,(-a\,r + t)^2}$

$\text{Riemann }[14,14] = -\dfrac{2}{9\,(-a\,r + t)^2}$

$$\text{Riemann}[14,41] = \frac{2}{9\,(-a\,r+t)^2}$$

$$\text{Riemann}[21,12] = -\frac{2\,(3\,a\,r+t)}{9\,(3\,a\,r-t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[21,21] = \frac{2\,(3\,a\,r+t)}{9\,(3\,a\,r-t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[23,23] = \frac{4\,t}{9\,(-3\,a\,r+t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[23,32] = -\frac{4\,t}{9\,(-3\,a\,r+t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[24,24] = \frac{4\,t}{9\,(-3\,a\,r+t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[24,42] = -\frac{4\,t}{9\,(-3\,a\,r+t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[31,13] = \frac{2}{(3\,a\,r-3\,t)^2}$$

$$\text{Riemann}[31,31] = -\frac{2}{(3\,a\,r-3\,t)^2}$$

$$\text{Riemann}[32,23] = -\frac{4\,t}{9\,(-3\,a\,r+t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[32,32] = \frac{4\,t}{9\,(-3\,a\,r+t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[34,34] = \frac{4}{9\,(-a\,r+t)^2}$$

$$\text{Riemann}[34,43] = -\frac{4}{9\,(-a\,r+t)^2}$$

$$\text{Riemann}[41,14] = \frac{2}{(3\,a\,r-3\,t)^2}$$

$$\text{Riemann}[41,41] = -\frac{2}{(3\,a\,r-3\,t)^2}$$

$$\text{Riemann}[42,24] = -\frac{4\,t}{9\,(-3\,a\,r+t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[42,42] = \frac{4\,t}{9\,(-3\,a\,r+t)\,(-a\,r+t)^2}$$

$$\text{Riemann}[43,34] = -\frac{4}{9\,(-a\,r+t)^2}$$

$$\text{Riemann}[43,43] = \frac{4}{9\,(-a\,r+t)^2}$$

RicciTensor[c, g]

Ricci Tensor [11] = $-\frac{2}{3(3a^2r^2-4art+t^2)}$

Ricci Tensor [12] = 0

Ricci Tensor [13] = 0

Ricci Tensor [14] = 0

Ricci Tensor [21] = 0

Ricci Tensor [22] = $\frac{2b^2(-3ar+t)}{27r^2(ar-t)(a-\frac{t}{r})^{2/3}}$

Ricci Tensor [23] = 0

Ricci Tensor [24] = 0

Ricci Tensor [31] = 0

Ricci Tensor [32] = 0

Ricci Tensor [33] = $-\frac{2b^2r(a-\frac{t}{r})^{1/3}}{9ar-3t}$

Ricci Tensor [34] = 0

Ricci Tensor [41] = 0

Ricci Tensor [42] = 0

Ricci Tensor [43] = 0

Ricci Tensor [44] = $-\frac{2b^2r(a-\frac{t}{r})^{1/3}\text{Sin}[\theta]^2}{9ar-3t}$

ScalarCurvature[c, g]

$$-\frac{4}{3(3a^2r^2-4art+t^2)}$$

KritchmanTensor[c, g]

KritchmanTensor [12,12] = $\frac{8(3ar+t)^2}{81(-3ar+t)^2(-ar+t)^4}$

KritchmanTensor [12,21] = $-\frac{8(3ar+t)^2}{81(-3ar+t)^2(-ar+t)^4}$

KritchmanTensor [13,13] = $\frac{8}{81(-ar+t)^4}$

KritchmanTensor [13,31] = $-\frac{8}{81(-ar+t)^4}$

KritchmanTensor [14,14] = $\frac{8}{81(-ar+t)^4}$

KritchmanTensor [14,41] = $-\frac{8}{81(-ar+t)^4}$

KritchmanTensor [21,12] = $-\frac{8(3ar+t)^2}{81(-3ar+t)^2(-ar+t)^4}$

KritchmanTensor [21,21] = $\frac{8 (3 a r + t)^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [23,23] = $\frac{32 t^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [23,32] = $-\frac{32 t^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [24,24] = $\frac{32 t^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [24,42] = $-\frac{32 t^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [31,13] = $-\frac{8}{81 (-a r + t)^4}$

KritchmanTensor [31,31] = $\frac{8}{81 (-a r + t)^4}$

KritchmanTensor [32,23] = $-\frac{32 t^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [32,32] = $\frac{32 t^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [34,34] = $\frac{32}{81 (-a r + t)^4}$

KritchmanTensor [34,43] = $-\frac{32}{81 (-a r + t)^4}$

KritchmanTensor [41,14] = $-\frac{8}{81 (-a r + t)^4}$

KritchmanTensor [41,41] = $\frac{8}{81 (-a r + t)^4}$

KritchmanTensor [42,24] = $-\frac{32 t^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [42,42] = $\frac{32 t^2}{81 (-3 a r + t)^2 (-a r + t)^4}$

KritchmanTensor [43,34] = $-\frac{32}{81 (-a r + t)^4}$

KritchmanTensor [43,43] = $\frac{32}{81 (-a r + t)^4}$

KritchmanScalar[c, g]

$$\frac{16 (21 a^2 r^2 - 10 a r t + 5 t^2)}{27 (-3 a r + t)^2 (-a r + t)^4}$$

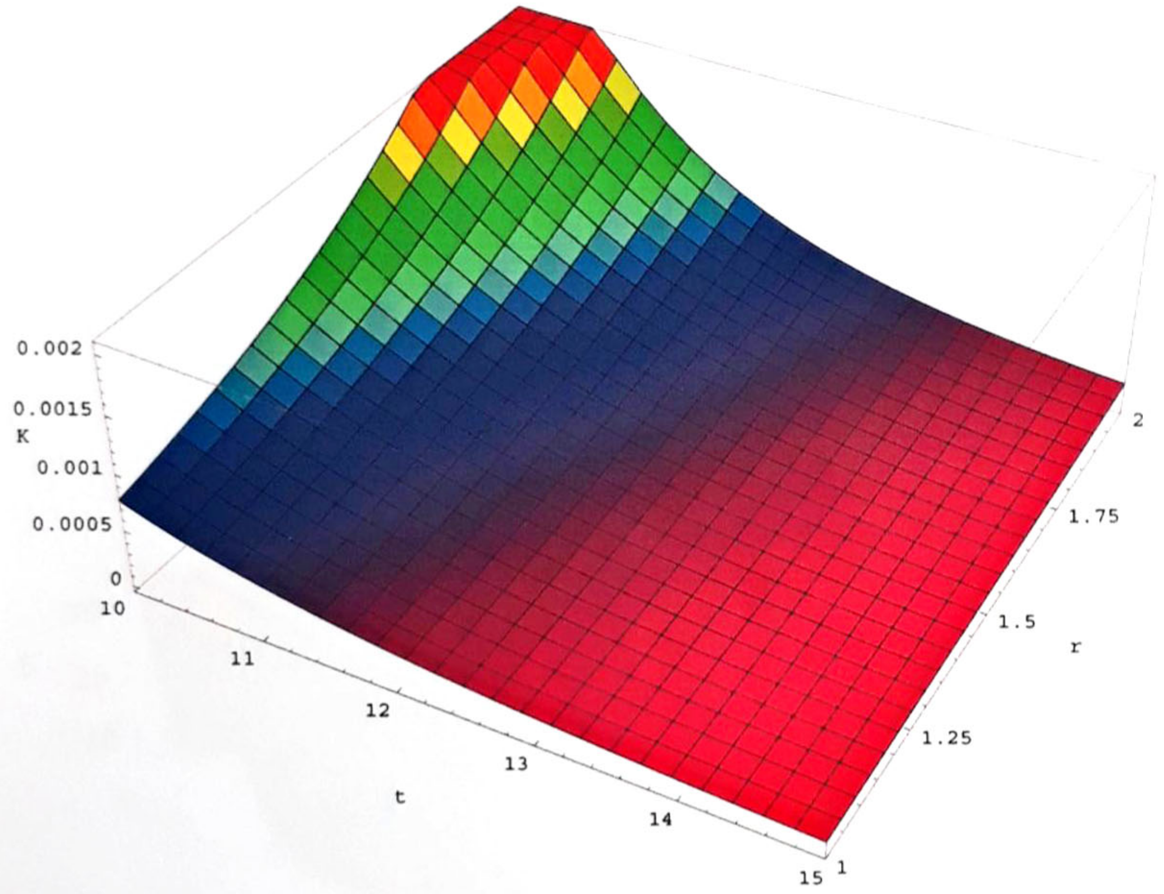

Figure 1.1: The profile of the Kretschmann scalar of the metric with an orthogonal time coordinate t and a radial area coordinate r

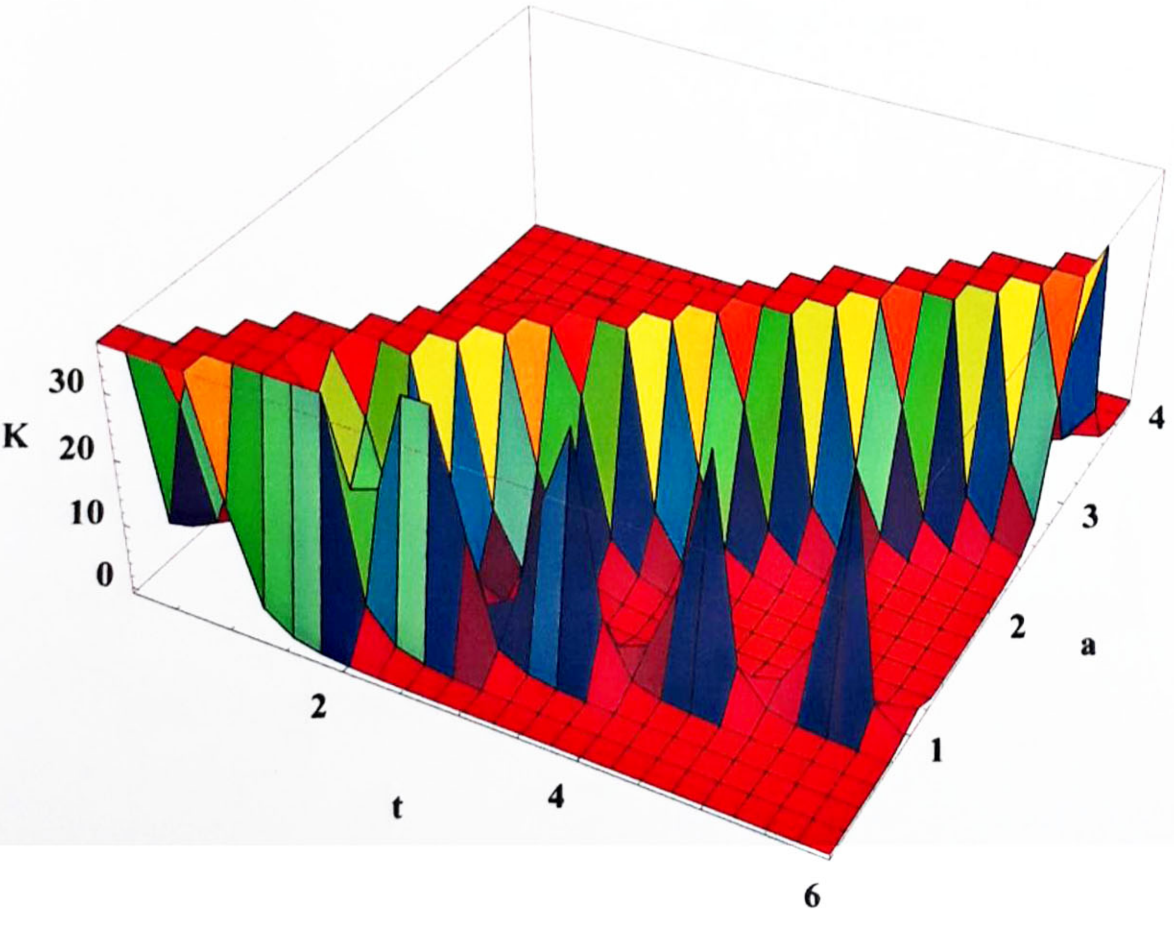

Figure 1.2: The profile of the Kretschmann scalar of the metric with an orthogonal time coordinate t and self-similar scale factor a.

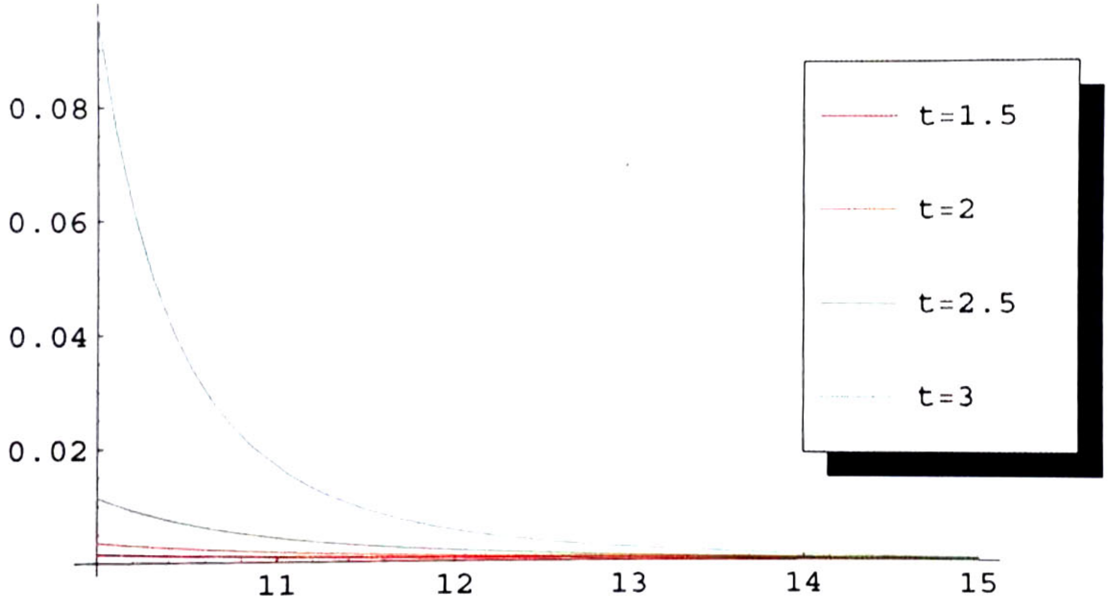

Figure 1.3: The variation of the Kretschmann scalar K of the metric with an orthogonal time coordinate t

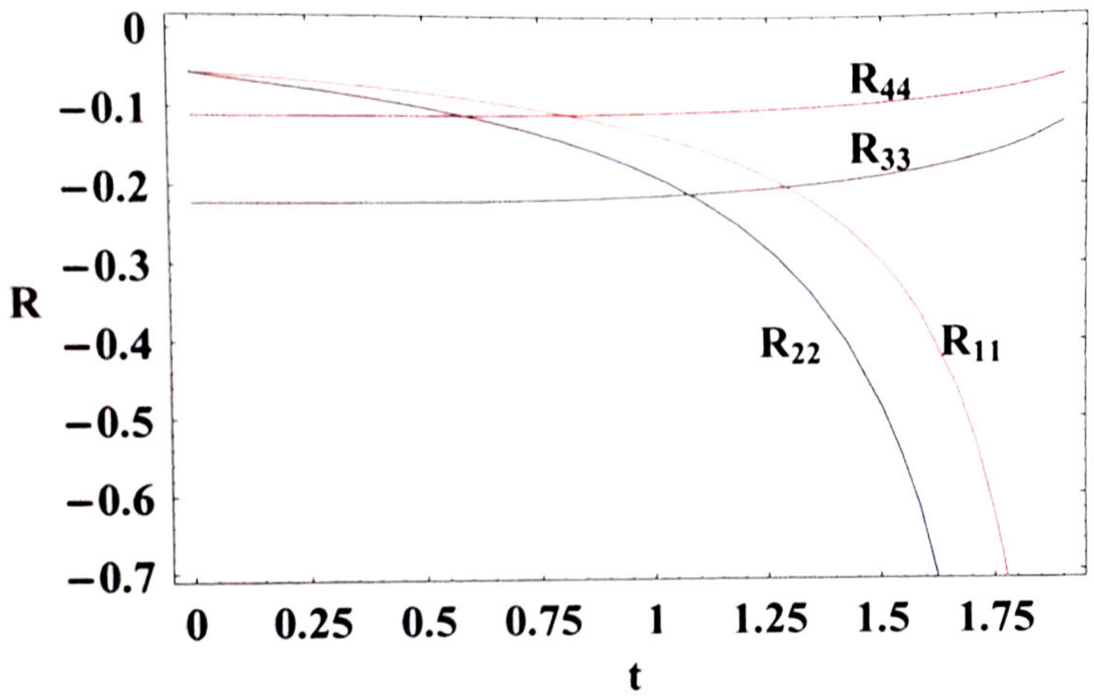

Figure 1.4: The variation of the Ricci tensor $R_{11}, R_{22}, R_{33}, R_{44}$ of the metric with an orthogonal time coordinate t

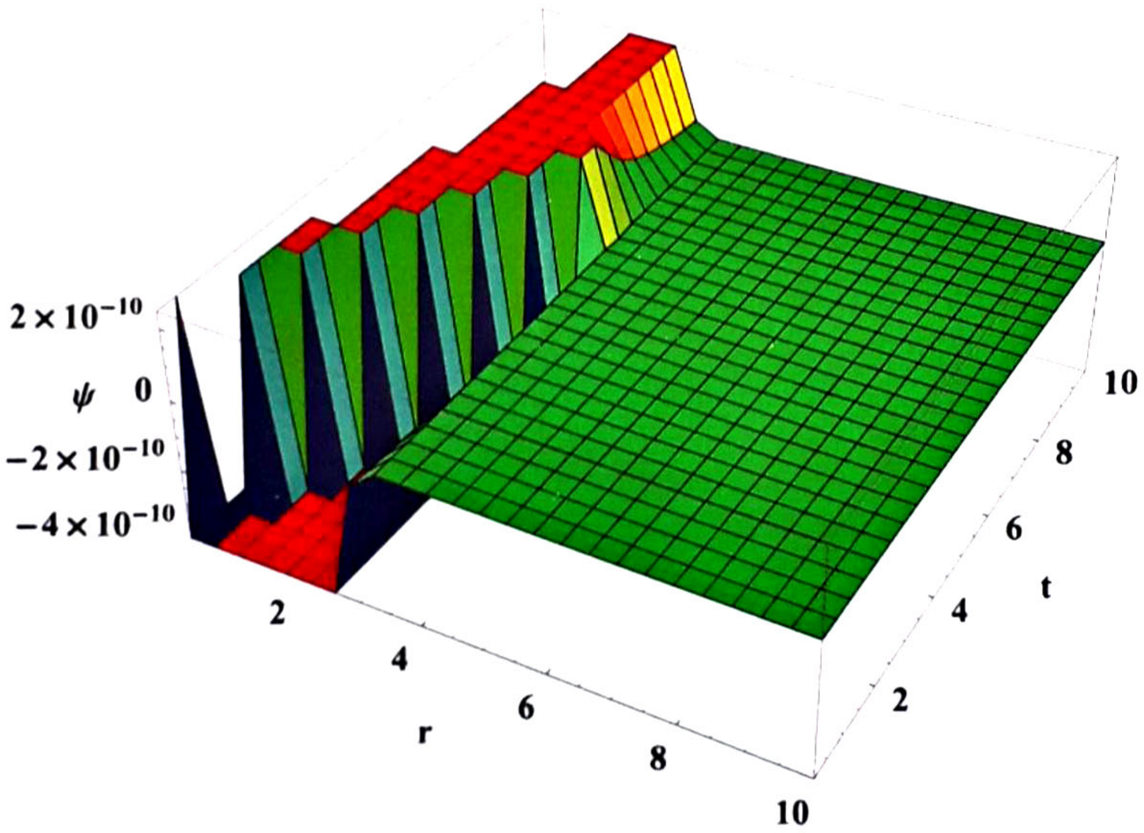

Figure 1.5: The profile of the curvature strength of the metric with an orthogonal time coordinate t and a radial area coordinate r

```
Clear[f, r, x]
f[r_, x_] = (-Log[-r + x])
h = 0.01;
r[0] = 0.5;
x[0] = 2;
r[i_] := r[i] = r[i - 1] + h
x[i_] := x[i] = x[i - 1] + h * f[r[i - 1], x[i - 1]]
euler = Table[{r[k], x[k]}, {k, 100}]
eulerplot = ListPlot[euler,
   PlotStyle → {PointSize[0.015], Red}, AxesLabel → {"R", "X"}, PlotJoined → False];
```

1.	1.91659
1.01	1.91746
1.02	1.91843
1.03	1.9195
1.04	1.92067
1.05	1.92194
1.06	1.92331
1.07	1.92478
1.08	1.92635
1.09	1.92802
1.1	1.92979
1.11	1.93165
1.12	1.93362
1.13	1.93568
1.14	1.93784
1.15	1.9401
1.16	1.94245
1.17	1.94491
1.18	1.94746
1.19	1.9501
1.2	1.95285
1.21	1.95569
1.22	1.95862
1.23	1.96165
1.24	1.96477
1.25	1.96799
1.26	1.97131
1.27	1.97471
1.28	1.97821
1.29	1.98181
1.3	1.98549
1.31	1.98927
1.32	1.99313
1.33	1.99709
1.34	2.00114
1.35	2.00528
1.36	2.0095
1.37	2.01382
1.38	2.01822
1.39	2.02271
1.4	2.02729
1.41	2.03195
1.42	2.0367
1.43	2.04154
1.44	2.04646
1.45	2.05146
1.46	2.05654
1.47	2.06171
1.48	2.06695
1.49	2.07228

0.51	1.99595
0.52	1.99198
0.53	1.98812
0.54	1.98435
0.55	1.98067
0.56	1.97709
0.57	1.9736
0.58	1.97021
0.59	1.96692
0.6	1.96372
0.61	1.96062
0.62	1.95761
0.63	1.9547
0.64	1.95189
0.65	1.94918
0.66	1.94656
0.67	1.94404
0.68	1.94162
0.69	1.93929
0.7	1.93707
0.71	1.93494
0.72	1.93291
0.73	1.93098
0.74	1.92915
0.75	1.92742
0.76	1.92578
0.77	1.92425
0.78	1.92282
0.79	1.92148
0.8	1.92025
0.81	1.91911
0.82	1.91807
0.83	1.91714
0.84	1.9163
0.85	1.91557
0.86	1.91493
0.87	1.9144
0.88	1.91396
0.89	1.91363
0.9	1.9134
0.91	1.91326
0.92	1.91323
0.93	1.9133
0.94	1.91347
0.95	1.91374
0.96	1.91411
0.97	1.91458
0.98	1.91515
0.99	1.91582
1	1.91659

(1.5 2.07769)

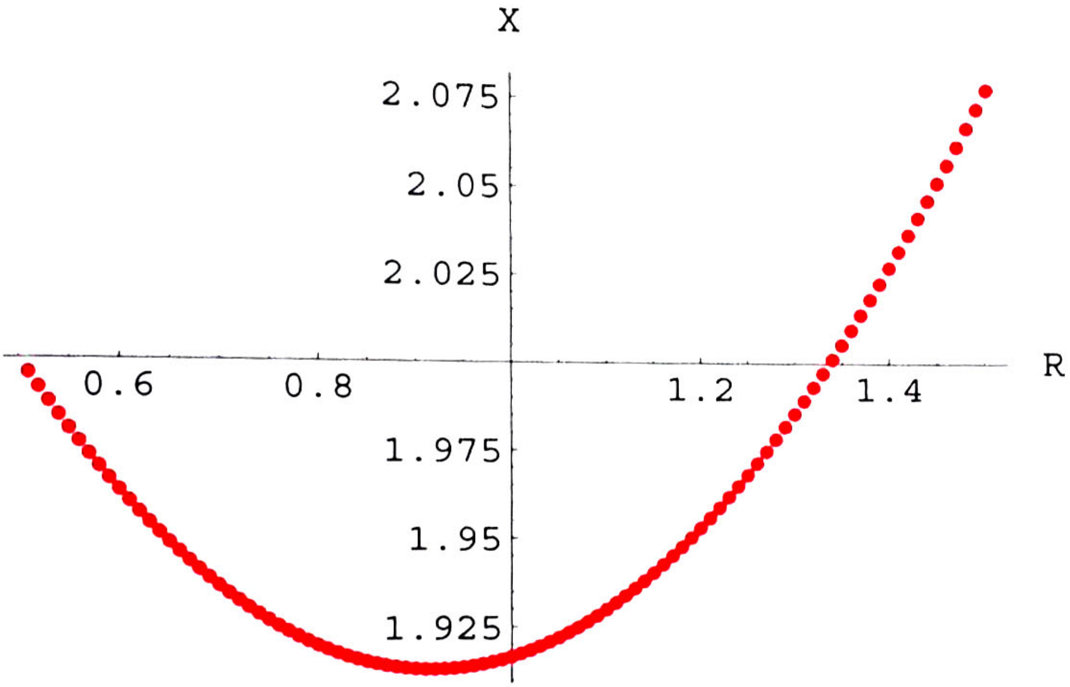

Figure 1.6: The iteration plot of the dimensionless parameters X and R (we set the initial values, $X = 2$ and $R = 0.5$)

Chapter 2

Self-Similarity in Newtonian Gravity

2.1 Introduction

Scale-invariance is one of the most fundamental characteristics of gravitational interaction in both Newtonian gravity and general relativity. This implies that if we consider appropriate matter fields, the governing partial differential equations are invariant under scale transformation. Due to this feature of the governing equations, there are self-similar solution, which are invariant under the scale transformation.

When a theory has no characteristic scale, we can expect scale-invariance of the theory. In Newtonian gravity, the gravitational constant G, with dimension $M^{-1}L^{3}T^{-2}$, is the only dimensional physical constant in the field equations, where M, L and T denote the dimension of mass, length and time, respectively. It is impossible to constant a physical scale only from G. In general relativity, there exists another physical constant c, which is the speed of light, with dimension LT^{-1}. In spite of these two dimensional constants, no characteristic length scale can be constructed from these physical constants. However, due to the existence of these two dimensional constants, general relativity is qualitatively different from Newtonian gravity with respect to scale invariance. If we consider quantum gravity, the Planck constant h appears, with dimension ML^2T^{-1}, so that there exists a characteristic scale $l_{pl} \equiv G^{\frac{1}{2}}h^{\frac{1}{2}}/c^{\frac{3}{2}}$,which is called the Planck length.

2.2 Self-Similarity in Newtonian Gravity

Since Newtonian gravity postulates an absolute system of space and time, we can directly apply the general formulation of self-similarity to this system (Baenblatt, 1996). A solution is called self-similar, if a dimensionless quantity $Z(t, \vec{x})$ made of the solution is of the form

$$Z(t, \vec{x}) = Z\left(\frac{\vec{x}}{a(t)}\right) \tag{2.1}$$

where $\vec{x}$ and t are independent space and time coordinates, respectively, and $a(t)$ is a function of t. This implies that the spatial distribution of the characteristics of motion remains similar to itself at all times during the motion. If the function $a(t)$ is derived from dimensional considerations alone, i.e., if it is uniquely determined so that $\vec{x}/a(t)$ is dimensionless, the self-similarity is called complete similarity or similarity of the first kind (Baenblatt, 1996).In more general situations, the characteristic length or time scale may be constructed by the dimensional constants in the system. Then, the function $a(t)$ cannot be uniquely determined from dimensional considerations alone. In such cases, self-similarity is called incomplete similarity or similarity of the second kind (Baenblatt, 1996). For example, when we have the constant sound speed c_s and no characteristic scale, then $a(t)$ is uniquely determined as $a(t) = c_s t$.In this case, the similarity is called complete. However, when we have a characteristic length scale l besides the sound speed c_s,then $a(t) = l^{1-\alpha}(c_s t)^\alpha$ is possible and the constant α may not be determined from the governing equations. In this case, the similarity is called complete. The constant α may be determined by the boundary conditions. It should be noted that the dimensional constant could appear not only from governing equations but also from boundary conditions.

Here, we give two important examples of completely self-similar solutions in Newtonian self-gravitating fluid mechanics. The basic field equations for spherically symmetric hydrodynamics of a self-gravitating ideal gas in Eulerian description are given by

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho, \tag{2.2}$$

$$\frac{\partial M}{\partial t} + v\frac{\partial M}{\partial r} = 0, \tag{2.3}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v^2) + \frac{\partial p}{\partial r} + \rho\frac{GM}{r^2} = 0, \quad (2.4)$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v^2) = 0, \quad (2.5)$$

where ρ, v, M and G denote the mass density, radial velocity, total mass inside the radial coordinate r, and gravitational constant, respectively.

2.3 Isothermal Gas

First we consider an isothermal gas as a gravitational source. Since the isothermal gas is a relevant description of cold molecular clouds in galaxies, self-similar solutions have been intensively studied in Newtonian gravity in modeling the star formation process (Hunter, 1977). It has been revealed that self-similar solutions play important roles in the gravitational collapse of an isothermal gas (Harada et al, 2003).

For an isothermal gas that obeys $p = c_s^2 \rho$, where c_s is the constant speed of sound with dimension LT^{-1}, it is possible to construct a characteristic scale from c_s and G. We introduce the dimensionless self-similar coordinate

$$z = \frac{c_s t}{r}, \quad (2.6)$$

for self-similar solutions. Then we also introduce the dimensionless functions U, P and m :

$$v(r,t) = -c_s U(r,t), \tag{2.7}$$

$$\rho(r,t) = \frac{c_s^2 P(r,t)}{4\pi G r^2}, \tag{2.8}$$

$$M(r,t) = \frac{c_s^3 t m(r,t)}{G}. \tag{2.9}$$

We assume that the above-defined functions U, P and m depend only on z. From this assumption, eqn (2.2) become

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho, \tag{2.10}$$

$$\frac{\partial M}{\partial r} - 4\pi r^2 \rho = 0,$$

$$\frac{\partial}{\partial r}\left(\frac{c_s^3 t m}{G}\right) - 4\pi r^2 \left(\frac{c_s^2 P}{4\pi G r^2}\right) = 0,$$

$$\frac{c_s^3 t}{G}\frac{\partial m}{\partial r} - \frac{c_s^2 P}{G} = 0,$$

$$\frac{\partial m}{\partial r} = \frac{P}{c_s t}, \tag{2.11}$$

By integrating equation(11) with respect to r, we get

$$m = \frac{Pr}{c_s t} = \frac{P}{z}, \tag{2.12}$$

$$\frac{dm}{dz} = \frac{d(\frac{P}{z})}{dz}, \tag{2.13}$$

$$m' = P(-z)^{-2}, \tag{2.14}$$

$$-z^2 m' = P, \tag{2.15}$$

In a similar manner, one can obtain,

$$m = P(U + \frac{1}{z}), \tag{2.16}$$

$$P' = \frac{zP[2 - P(zU+1)]}{(zU+1)^2 - z^2}, \tag{2.17}$$

$$U' = \frac{(zU+1)[P(zU+1) - 2]}{(zU+1)^2 - z^2}, \tag{2.18}$$

where the prime denotes the derivation with respect to z. The self-similar solutions for an isothermal gas are obtained from these ordinary differential equations. Self-similar solutions scale for the scale transformations $\bar{t} = at$, $\bar{r} = ar$, as

$$v(\bar{r}, \bar{t}) = v(r, t), \tag{2.19}$$

$$\rho(\bar{r}, \bar{t}) = \frac{\rho(r,t)}{a^2}, \tag{2.20}$$

$$M(\bar{r}, \bar{t}) = aM(r, t), \tag{2.21}$$

where a is a constant. The basic equations for self-similar solutions are singular at the center and at the point at which $(zU + 1)^2 - z^2 = 0$ is satisfied, which is called a sonic point.

2.4 Polytropic Gas

Next, we consider a polytropic gas as a gravitational source. A polytropic gas obeys the equation of state $p = K\rho^\gamma$, where γ is the dimensionless constant called the adiabatic exponent and K is a constant with dimension

$M^{1-\gamma}L^{3\gamma-1}T^{-2}$. As in the isothermal gas system, it is impossible to construct a characteristic scale only from G and K if $\gamma \neq 2$. For the exceptional case, $\gamma = 2$, the system has a characteristic length scale $l = \sqrt{K/G}$ but even in this case the self-similar variable z is uniquely constructed. Then, complete similarity is applicable to this system.

For the polytropic case, we introduce the dimensionless self-similar coordinate

$$z = \frac{\sqrt{K}(-t)^{2-\gamma}}{(4\pi G)^{\frac{(\gamma-1)}{2}} r}. \tag{2.22}$$

Then we also introduce the dimensionless functions U, P and m :

$$\begin{aligned} v(r,t) &= -(4\pi G)^{\frac{(1-\gamma)}{2}} \sqrt{K}(-t)^{1-\gamma} U(r,t), && (2.23) \\ \rho(r,t) &= \frac{K^{\frac{1}{(2-\gamma)}} P(r,t)}{(4\pi G)^{\frac{1}{(2-\gamma)}} r^{\frac{2}{(2-\gamma)}}}, && (2.24) \\ M(r,t) &= \frac{K^{\frac{3}{2}}(-t)^{4-3\gamma} m(r,t)}{(4\pi)^{\frac{3(\gamma-1)}{2}} G^{\frac{(3\gamma-1)}{2}}}. && (2.25) \end{aligned}$$

We assume that the above-defined functions U, P and m depend only on z. In the polytropic case, the sonic point is defined by $(2-\gamma-zU)^2 - \gamma z^{\frac{2}{(2-\gamma)}} = 0$. Self-similar solutions scale for the scale transformations $\bar{t} = at$, $\bar{r} = ar$, as

$$\begin{aligned} v(\bar{r},\bar{t}) &= a^{1-\gamma} v(r,t), && (2.26) \\ \rho(\bar{r},\bar{t}) &= \frac{\rho(r,t)}{a^{\frac{2}{(2-\gamma)}}}, && (2.27) \\ M(\bar{r},\bar{t}) &= a^{4-3\gamma} M(r,t), && (2.28) \end{aligned}$$

where a is a constant. In this case, the scaling rates for r and t, which keep z constant, are different from each other.

It should be again emphasized that in both the isothermal and polytropic cases, the self-similarity is complete since the self-similar variable z can be obtained from dimensional considerations alone. This is because there are only two dimensional constants in the system, while there are three independent dimensions M, L and T.

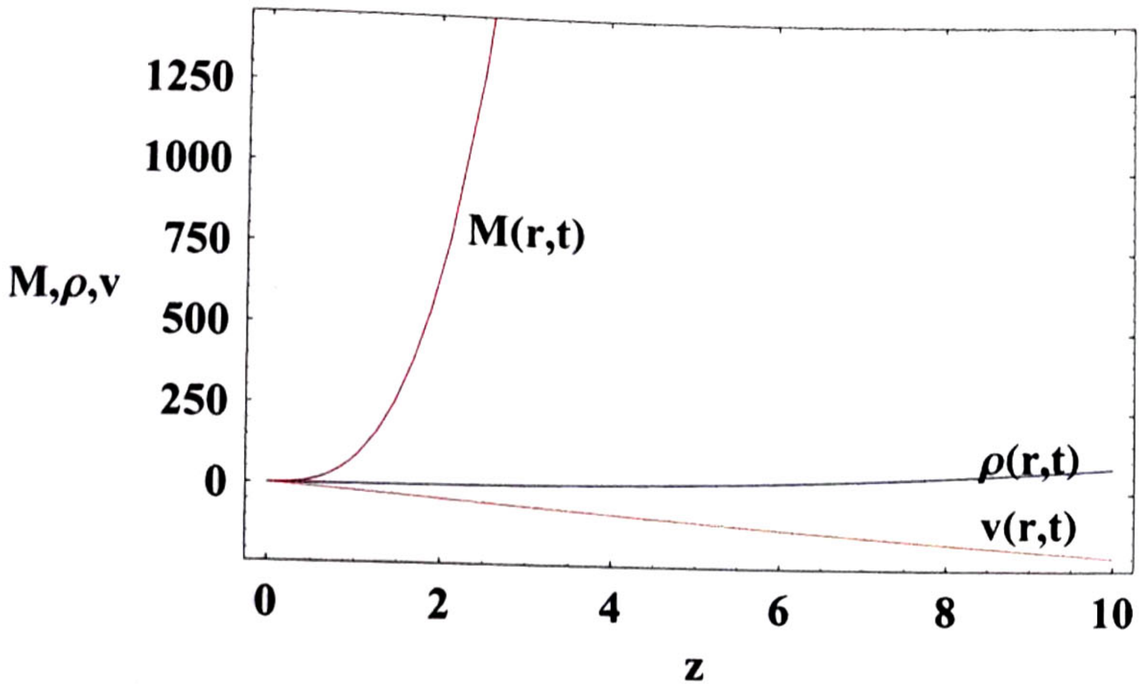

Figure 2.1: The comparison of the self-similar functions of the mass density ρ, radial velocity v and total mass M inside the radial coordinate r with varying self-similar coordinate z of isothermal gas

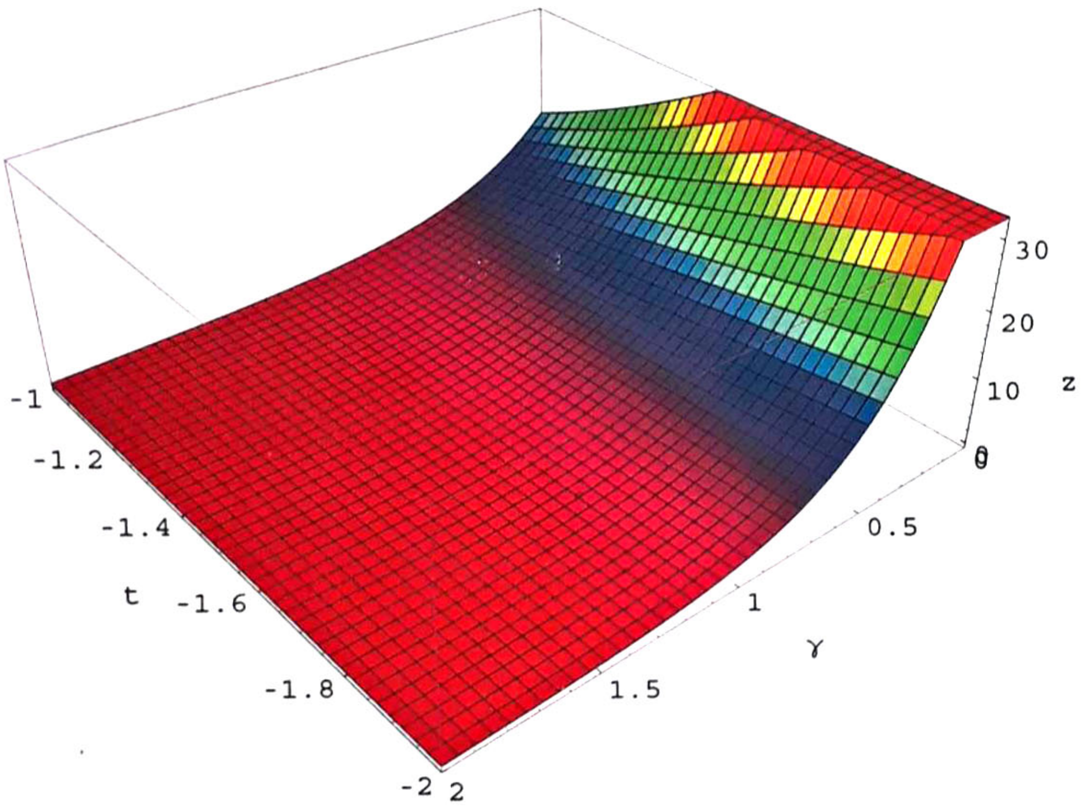

Figure 2.2: The profile of dimensionless self-similar coordinate z of polytropic gas. with an orthogonal time coordinate t and a radial area coordinate r

```
ClearAll["Global`*"]

Clear[f, m, a]
f[a_, m_] = (a)^(4-(3×2)) × m
```

$$\frac{m}{a^2}$$

```
h = 1;
a[0] = 2;
m[0] = .5;

a[i_] := a[i] = a[i - 1] + h
m[i_] := m[i] = m[i - 1] + h * f[a[i - 1], m[i - 1]]

euler = Table[{a[k], m[k]}, {k, 50}]
eulerplot = ListPlot[euler, PlotStyle → {PointSize[0.02], Red},
  AxesLabel → {"a", "m"}, PlotJoined → False, PlotRange → {.60, .90}]
```

3	0.625
4	0.694444
5	0.737847
6	0.767361
7	0.788677
8	0.804772
9	0.817347
10	0.827437
11	0.835712
12	0.842618
13	0.84847
14	0.853491
15	0.857845
16	0.861658
17	0.865024
18	0.868017
19	0.870696
20	0.873108
21	0.87529
22	0.877275
23	0.879088
24	0.88075
25	0.882279
26	0.88369
27	0.884998
28	0.886212
29	0.887342
30	0.888397
31	0.889384
32	0.89031
33	0.891179
34	0.891997
35	0.892769
36	0.893498
37	0.894187
38	0.89484
39	0.89546
40	0.896049
41	0.896609
42	0.897142
43	0.897651
44	0.898136
45	0.8986
46	0.899044
47	0.899469
48	0.899876
49	0.900267
50	0.900642
51	0.901002
52	0.901348

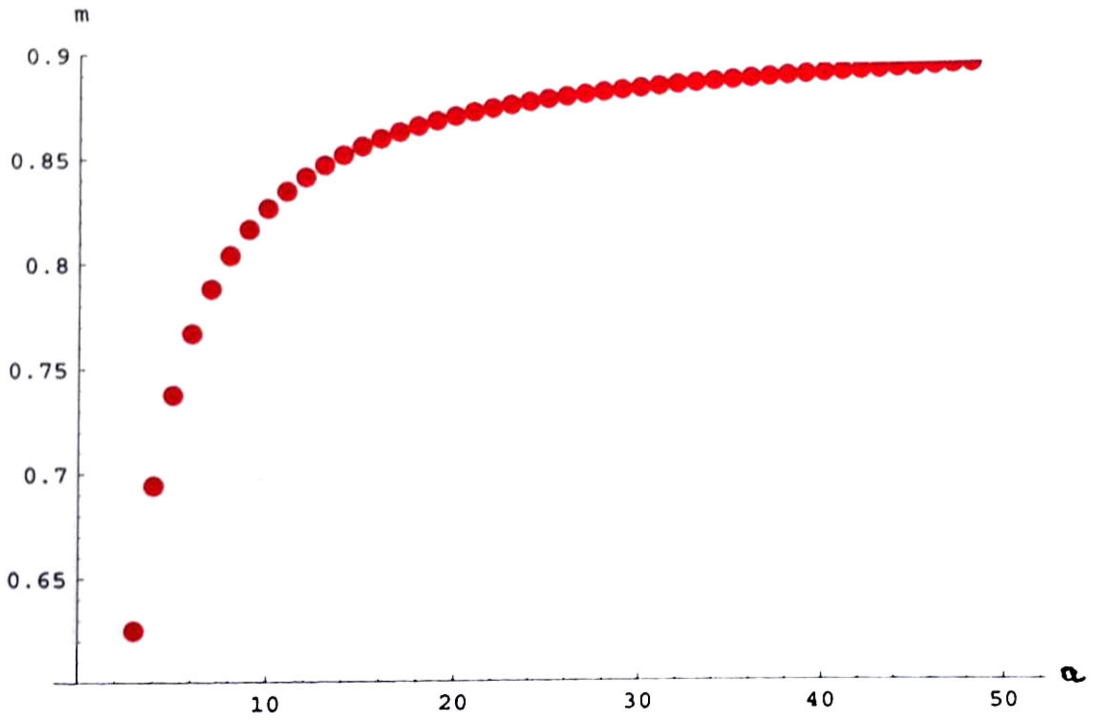

Figure 2.3: Iteration plot of the self-similar mass function of the metric(we set the initial values $a = 2$ and $m = 0.5$)

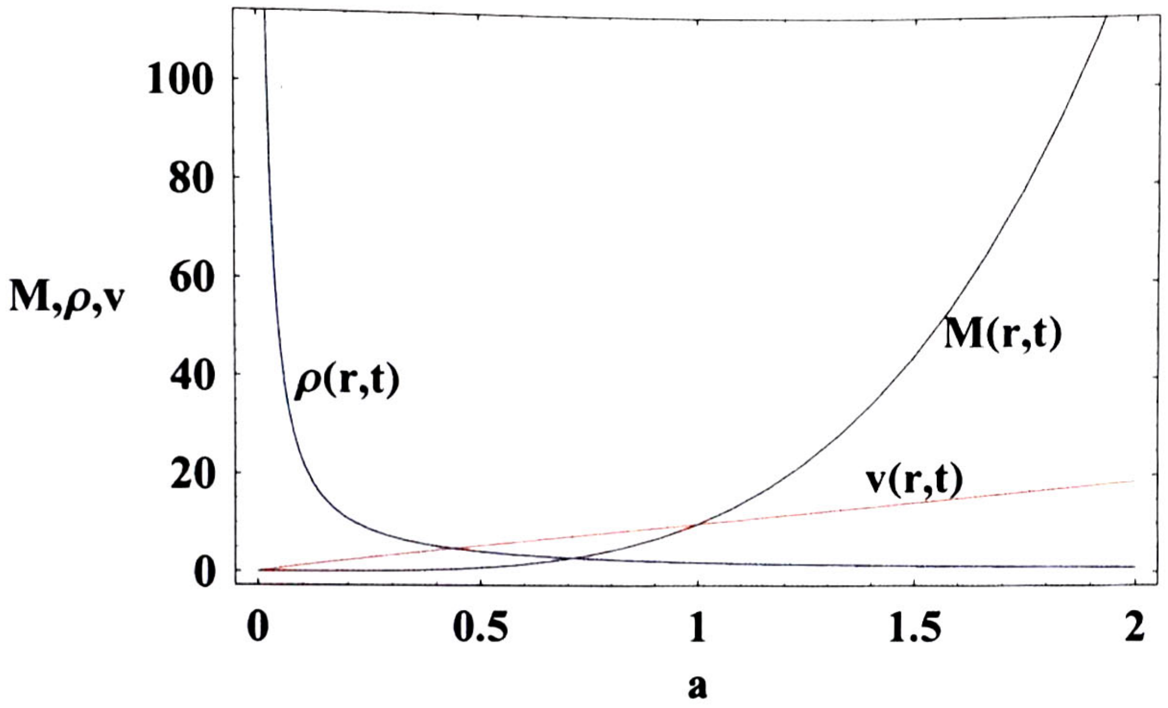

Figure 2.4: The comparison of the self-similar functions of the mass density ρ, radial velocity v and total mass M inside the radial coordinate r with varying self-similar dimensionless constant a of polytropic gas

Chapter 3

Self-Similarity in General Relativity

3.1 Introduction

In general relativity, the concept of self-similarity is not so straightforward because general relativity has general covariance against coordinate transformation. This implies that the definition should be made covariantly in general relativity. In the following, we use units where the speed of light c is unity. In this choice of units, $T = L$ is obtained and the velocity is dimensionless.

In general relativity, the term self-similarity can be used in two ways. One is for the properties of spacetimes, the other is for the properties of matter fields. These are not equivalent in general. The self-similarity in general relativity was defined for the first times by (Cahill and Taub, 1971). Self-similarity is defined by the existence of a homothetic vector ξ in the spacetime, which satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = 2\alpha g_{\mu\nu}, \tag{3.1}$$

where $g_{\mu\nu}$ is the metric tensor, $\mathcal{L}_\xi$ denotes Lie differentiation along ξ and α is a constant. This is a special type of conformal Killing vectors. This self-similarity is called homothety. If $\alpha \neq 0$, then it can be set to be unity by a constant rescaling of ξ. If $\alpha = 0, i.e, \mathcal{L}_\xi g_{\mu\nu} = 0$, then ξ is a Killing vector. Homothety is a purely geometric property of spacetime so that the physical quantity does not necessarily exhibit self-similarity such as $\mathcal{L}_\xi Z = dZ$, where d is a constant and Z is, for example, the pressure, the energy density and so on. From equation (3.1) it follows that

$$\mathcal{L}_\xi R^\mu_{\nu\sigma\rho} = 0, \tag{3.2}$$

and hence

$$\mathcal{L}_\xi R_{\mu\nu} = 0, \tag{3.3}$$

$$\mathcal{L}_\xi G_{\mu\nu} = 0, \tag{3.4}$$

A vector field ξ that satisfies equations (3.2),(3.3) and (3.4) is called a curvature collineation, a Ricci collineation and a matter collineation, respectively. It is noted that equations (3.2), (3.3) and (3.4) do not necessarily mean that ξ is a homothetic vector. We consider the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.5}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. If the spacetime is homothetic, the energy-momentum tensor of the matter fields must satisfy

$$\mathcal{L}_\xi T_{\mu\nu} = 0, \tag{3.6}$$

through eqns (3.5) and (3.4). For a perfect fluid case, the energy-momentum tensor takes the form of

$$T_{\mu\nu} = (p + \mu)u_\mu u_\nu + p g_{\mu\nu}, \tag{3.7}$$

where p and μ are the pressure and the energy density, respectively. Then, eqns (3.1) and (3.6) result in

$$\mathcal{L}_\xi u^\mu = -\alpha u^\mu, \tag{3.8}$$

$$\mathcal{L}_\xi \mu = -2\alpha\mu, \tag{3.9}$$

$$\mathcal{L}_\xi p = -2\alpha p. \tag{3.10}$$

As shown above, for a perfect fluid, the self-similarity of the spacetime and that of the physical quantity coincide. However, this fact does not necessarily hold for more general matter fields.

3.2 Kinematic Self-Similarity

Although homothetic solutions can contain several interesting matter fields, the matter fields compatible with homothety are rather limited. In more general situations, matter fields will have intrinsic dimensional constants. For example, when we consider a polytropic equation of state, such as $p = K\mu^{\gamma}$, the constant K has dimension $M^{1-\gamma}L^{3(\gamma-1)}$, where we should be reminded that we have chosen the light speed c to be unity. We can also consider a massive scalar field, where the mass of the scalar field has dimension M. In such cases, it is impossible to assume homothety because the system has a characteristic scale. By analogy, we can consider the general relativistic counterpart of incomplete similarity (Coley, 1997). From comparison with self-similarity for a polytropic gas in Newtonian gravity, kinematic self-similarity has been defined in the context of relativistic fluid mechanics as an example of incomplete similarity. It should be noted that the introduction of incomplete similarity to general relativity is not unique. For example, partial self-similarity has been defined and applied to inhomogeneous cosmological solutions.

A spacetime is said to be kinematic self-similar if it admits a kinematic self-similar vector ξ which satisfies the conditions

$$\mathcal{L}_{\xi} h_{\mu\nu} = 2\delta h_{\mu\nu}, \tag{3.11}$$

$$\mathcal{L}_{\xi} u_{\mu} = \alpha u_{\mu}, \tag{3.12}$$

where u^{μ} is the four-velocity of the fluid and $h_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$ is the projection tensor, and α and δ are constants (Coley, 1997). If $\delta \neq 0$, the similarity transformation is characterized by the scale independent ratio α/δ, which is referred to as the similarity index. If the ratio is unity, ξ turns out to be a homothetic vector. In the context of kinematic self-similarity, homothety is referred to as self-similarity of the first kind. If $\alpha = 0$ and $\delta \neq 0$, it is referred to as self-similarity of the zeroth kind. If the ratio is not equal to zero or one, it is referred to as self-similarity of the second kind. If $\alpha \neq 0$ and $\delta = 0$, it is referred to as self-similarity of the infinite kind. If $\delta = \alpha = 0$, ξ turns out to be a Killing vector.

From the Einstein eqn (3.5), we can derive

$$\mathcal{L}_\xi G_{\mu\nu} = 8\pi G \mathcal{L}_\xi T_{\mu\nu} \tag{3.13}$$

This equation is called the integrability condition. Now we can rewrite the integrability conditions (3.13) in terms of kinematic quantities of the fluid. The covariant derivative of the fluid four velocity is decomposed into the following form:

$$u_{\mu;\nu} = \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} + \omega_{\mu\nu} - \dot{u}_\mu u_\nu \tag{3.14}$$

where

$$\theta_{\mu\nu} \equiv h^\kappa_{(\mu} h^\lambda_{\nu)} u_{\kappa;\lambda}, \tag{3.15}$$

$$\theta \equiv g^{\mu\nu}\theta_{\mu\nu}, \tag{3.16}$$

$$\sigma_{\mu\nu} \equiv \theta_{\mu\nu} - \frac{1}{3}\theta h_{\mu\nu}, \tag{3.17}$$

$$\omega_{\mu\nu} \equiv h^\kappa_{[\mu} h^\lambda_{\nu]} u_{\kappa;\lambda}, \tag{3.18}$$

$$\omega^2 \equiv \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}, \tag{3.19}$$

$$\dot{u}_\mu \equiv u_{\mu;\nu}u^\nu, \tag{3.20}$$

where the semicolon denotes the covariant derivative. Using the above quantities the integrability condition (3.13) is rewritten as follows

$$(\delta-\alpha)(-8\omega^2-2\dot{u}^{;\kappa}_{\kappa}) = 8\pi G\left[\frac{1}{2}(\mathcal{L}_\xi\mu+2\alpha\mu)+\frac{3}{2}(\mathcal{L}_\xi p+2\alpha p)\right], \quad (3.21)$$

$$2(\delta-\alpha)(\dot{\theta}+\theta^2-4\omega^2) = 8\pi G\left[\frac{3}{2}(\mathcal{L}_\xi\mu+2\delta\mu)-\frac{3}{2}(\mathcal{L}_\xi p+2\delta p)\right], \quad (3.22)$$

$$2\omega_{\lambda\mu}\dot{u}^\mu+2\omega^{;\kappa}_{\kappa\lambda}-4\omega^2 u_\lambda=0, \quad (3.23)$$

$$\dot{\sigma}_{\lambda\rho}-u_\rho\sigma_{\lambda\nu}\dot{u}^\nu-u_\lambda\sigma_{\rho\mu}\dot{u}^\mu+\theta\sigma_{\lambda\rho}+\sigma_{\lambda\kappa}\omega^\kappa_\rho+\sigma_{\rho\kappa}\omega^\kappa_\lambda+2\omega^\kappa_\lambda\omega_{\kappa\rho}+\frac{4}{3}h_{\lambda\rho}\omega^2=0. \quad (3.24)$$

For the first-kind case, in which $\alpha=\delta\neq 0$, eqns (3.9) and (3.10) are obtained from eqns (3.21) and (3.22).When a perfect fluid is irrotational, i.e., $\omega_{\mu\nu}=0$,the Einstein equations and the integrability conditions (3.21) - (3.24) give (Collay , 1997)

$$(\alpha-\delta)R_{\mu\nu}=0, \quad (3.25)$$

where $R_{\mu\nu}$ is the Ricci tensor on the hypersurface orthogonal to u^μ. This means that if a solution is kinematic self-similar but not homothetic and if the fluid is irrotational, then the hypersurface orthogonal to fluid flow is flat.

3.3 Spherically Symmetric Self-Similar Solutions

Although self-similar solutions can play important roles even in nonspherically symmetric solution, such as homogeneous cosmological models (Wainwright, 1990), we focus in the rest of this section on spherically symmetric spacetimes. The line element in a spherically symmetric spacetime is given by

$$ds^2 = -e^{2\phi(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R(t,r)^2 d\Omega^2, \tag{3.26}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. We consider a perfect fluid as a matter field, for which the energy-momentum tensor is given by eqn (3.7). We adopt the comoving coordinates, where the four-velocity of the fluid u^μ has the components

$$u_\mu = (-e^\phi, 0, 0, 0). \tag{3.27}$$

Then, the Einstein equations and the equations of motion for the perfect fluid are reduced to the following simple form:

$$(\mu + p)\phi_r = -p_r, \tag{3.28}$$

$$(\mu + p)\psi_t = -\mu_t - 2(\mu + p)\frac{R_t}{R}, \tag{3.29}$$

$$m_r = 4\pi\mu R_r R^2, \tag{3.30}$$

$$m_t = -4\pi p R_t R^2, \tag{3.31}$$

$$0 = -R_{tr} + \phi_r R_t + \psi_t R_r, \tag{3.32}$$

$$2Gm = R(1 + e^{-2\psi}R_t^2 - e^{-2\psi}R_r^2), \tag{3.33}$$

where the subscripts t and r denote the partial derivatives with respect to t and r, respectively, and $m(t,r)$ is called Misner-Sharp mass. When a perfect fluid obeys an equation state $p + \mu = 0$, which is equivalent to a cosmological constant, the first two equations are trivially satisfied. In this case, one can use the following equations:

$$-\frac{e^{2\phi}}{R^2}-\left[(\frac{R_t}{R})^2+2\frac{R_t}{R}\psi_t\right]+e^{2\phi-2\psi}\left[2\frac{R_{rr}}{R}-2\frac{R_r}{R}\psi_r+\left(\frac{R_r}{R}\right)^2\right]=-8\pi G\mu e^{2\phi}, \tag{3.34}$$

$$\frac{e^{2\psi}}{R^2}+e^{2\psi-2\phi}\left[2\frac{R_{tt}}{R}-2\frac{R_t}{R}\phi_t+\left(\frac{R_t}{R}\right)^2\right]-\left[(\frac{R_r}{R})^2+2\frac{R_r}{R}\phi_r\right]=-8\pi Gpe^{2\psi}, \tag{3.35}$$

$$e^{-2\phi}\left(\psi_{tt}+\psi_t^2-\phi_t\psi_t+\frac{R_{tt}}{R}+\frac{R_t\psi_t}{R}-\frac{R_t\phi_T}{R}\right)$$
$$-e^{-2\psi}\left(\phi_{rr}+\phi_r^2-\phi_r\psi_r+\frac{R_{rr}}{R}+\frac{R_r\phi_r}{R}-\frac{R_r\psi_r}{R}\right)=-8\pi Gp, \tag{3.36}$$

which are (tt), (rr) and $(\theta\theta)$ component of the Einstein equations, respectively. Five of the above nine equations are independent.

3.4 Spherically Symmetric Homothetic Solutions

There is a large variety of spherically symmetric homothetic solutions. The pioneering work in this area was done by Cahill and Taub (Cahill and Taub, 1971). The application contains primordial black holes (Hawking, 1974), cosmological voids, cosmic censorship (Lake, 1992) and critical behavior (Chaptuik, 1993), respectively. The classification of all spherically symmetric homothetic solutions with a perfect fluid has been made by Carr (Carr, 2000).. The spacetime structure possible for homothetic solutions has been studied by Carr (Carr, 2003). The special case where the homothetic vector is orthogonal or parallel to the fluid flow has also been studied by Coley (Coley, 1991). It has been revealed that a homothetic solution describes the dynamical properties of more general solutions in spherically symmetric gravitational collapse. The stability of homothetic solutions has been studied by Harada (Harada,. 2001).

When the homothetic admits a homothetic vector, which is neither parallel nor orthogonal to the fluid flow, the homothetic vector ξ can be written as

$$\xi = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \tag{3.37}$$

and the self-similar variable ξ is given by

$$\xi = \frac{r}{t} \tag{3.38}$$

Homothety implies that the metric functions can be written

$$ds^2 = -e^{2\phi(\xi)}dt^2 + e^{2\psi(\xi)}dr^2 + r^2 S(\xi)^2 d\Omega^2. \tag{3.39}$$

As we have seen, the equation of state must be of the form $p = K\mu$ for homothetic spacetimes. Then the governing equations for homothetic solutions are written as

$$e^{2\phi} = a_\sigma \xi^{\frac{4K}{1+K}} \eta^{-\frac{2K}{1+K}}, \tag{3.40}$$

$$e^{2\psi} = a_\omega \eta^{-\frac{2}{1+K}} S^{-4}, \tag{3.41}$$

$$M + M' = \eta S^2 (S + S'), \tag{3.42}$$

$$M' = -K\eta S^2 S', \tag{3.43}$$

$$\frac{M}{S} = 1 + a_\sigma^{-1} (\eta \xi^{-2})^{\frac{2K}{1+K}} \xi^2 S'^2 - \eta^{\frac{2}{1+K}} S^4 (S + S')^2, \tag{3.44}$$

where a_σ and a_ω are integration constants and the prime denotes the derivative with respect to $ln\xi$. The dimensionless functions $\eta(\xi)$ and $M(\xi)$ are defined by

$$8\pi G\mu = \frac{\eta}{r^2}, \tag{3.45}$$

$$2Gm = rM. \tag{3.46}$$

The above formulation is based on Cahill and Taub (Cahill and Taub, 1971).It is possible to choose another function in the same comoving coordinates, as adopted in Carr (Carr, 2000).In the comoving coordinates, the dynamical properties of the fluid elements are very clear.

There are other useful formulations in analyzing homothetic solutions. One of the most natural coordinate systems for homothetic spacetimes is the so-called homothetic coordinates.In terms of this coordinate system, the dynamical systems theory has been applied to homothetic solutions with a perfect fluid for classification. In the homothetic coordinates, the self-similar variable is chosen to be the spatial or time coordinate, depending on whether the homothetic vector is timelike or spacelike. If the homothetic vector is timelike, the line element is written as

$$ds^2 = e^{2t}[-D_1^2(x)dt^2 + dx^2 + D_2^2(x)d\Omega^2]. \tag{3.47}$$

If the homothetic vector is spacelike, the line element is written as

$$ds^2 = e^{2x}[dt^2 + -D_1^2(t)dx^2 + D_2^2(t)d\Omega^2]. \tag{3.48}$$

If the homothetic vector is timelike in one region and spacelike in another region of the same spacetime, the above two charts must be patched on the hypersurface on which the homothetic vector is null.

Another coordinate system is that of area coordinates, in which the physical properties of the spacetime are clear. The area coordinate system has been adopted by Ori(Ori, 1990). In this coordinate system, the line element in homothetic spacetimes is written as

$$ds^2 = -e^{2\bar{\Phi}(z)}dt^2 + e2\bar{\Psi}(z)dr^2 + r^2 d\Omega^2, \quad (3.49)$$

$$z = \frac{r}{t}, \quad (3.50)$$

$$u^2 \frac{\partial}{\partial x^\mu} = u^t(z)\frac{\partial}{\partial t} + u^r(z)\frac{\partial}{\partial r}, \quad (3.51)$$

where u^t and u^r are also to be determined.

3.4.1 Spherically Symmetric Kinematic Self-Similar Solutions

A kinematic self-similar vector may be parallel, orthogonal or tilted, i.e., neither parallel nor orthogonal, to the fluid flow. Spherically symmetric kinematic self-similar perfect fluid solutions have been recently explored by several authors.

In a spherically symmetric spacetime, the kinematic self-similar vector field ξ is written in general as

$$\xi = h_1(t,r)\frac{\partial}{\partial t} + h_2(t,r)\frac{\partial}{\partial r}, \quad (3.52)$$

in the comoving coordinates, where $h_1(t,r)$ and $h_2(t,r)$ are functions of t and r. When $h_2 = 0, \xi$ is parallel to the fluid flow, while when $h_1 = 0, \xi$ is orthogonal to the fluid flow. When both h_1 and h_2 are nonzero, ξ is titled. When the kinematic self-similar vector ξ is titled to the fluid flow and not of the infinite kind, ξ and the metric tensor $g_{\mu\nu}$ are written in appropriate comoving coordinates as

$$\xi = (\alpha t + \beta)\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \tag{3.53}$$

$$ds^2 = -e^{2\Phi(\xi)}dt^2 + e^{2\Psi(\xi)}dr^2 + r^2 S(\xi)^2 d\Omega^2, \tag{3.54}$$

where α is the index of self-similarity. For $\alpha = 1$, i.e., homothety or self-similarity of the first kind, we can set $\beta = 0$ and then ξ is given by $\xi = r/t$. For $\alpha = 0$, i.e., self-similarity of the zeroth kind, we can set $\beta = 1$ and then ξ is given by $\xi = r/e^t$. For $\alpha \neq 0$ and $\alpha \neq 1$,i.e., self-similarity of the second kind, we can set $\beta = 0$and then ξ is given by $\xi = r/(\alpha t)^{1/\alpha}$. If the kinematic self-similar vector ξ is titled to the fluid flow and of the infinite kind, ξ and the metric tensor $g_{\mu\nu}$ are written in appropriate comoving coordinates as

$$\xi = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \tag{3.55}$$

$$ds^2 = -e^{2\Phi(\xi)}dt^2 + \frac{e^{2\Psi(\xi)}}{r^2}dr^2 + S(\xi)^2 d\Omega^2, \tag{3.56}$$

where the self-similar variable is given by $\xi = r/t$. If the kinematic self-similar vector ξ is parallel to the fluid flow and not the infinite kind, we have in appropriate comoving coordinates as

$$\xi = t\frac{\partial}{\partial t}, \tag{3.57}$$

$$ds^2 = -t^{2(\alpha-1)}e^{2\Phi(r)}dt^2 + t^2dr^2 + t^2S(r)^2d\Omega^2, \tag{3.58}$$

where α is the index of self-similarity and the self-similar variable is given by $\xi = r$. If the kinematic self-similar vector ξ is parallel to the fluid flow and of the infinite kind, we have in appropriate comoving coordinates as

$$\xi = t\frac{\partial}{\partial t}, \tag{3.59}$$

$$ds^2 = -e^{2\Phi(r)}dt^2 + dr^2 + S(r)^2d\Omega^2, \tag{3.60}$$

where the self-similar variable is given by $\xi = r$. If the kinematic self-similar vector ξ is orthogonal to the fluid flow and not of the infinite kind, we have in appropriate coordinates

$$\xi = r\frac{\partial}{\partial r}, \tag{3.61}$$

$$ds^2 = -r^{2\alpha}dt^2 + e^{2\Psi(t)}dr^2 + r^2S(t)^2d\Omega^2, \tag{3.62}$$

where α is the index of self-similarity and the self-similar variable is given by $\xi = t$. If the kinematic self-similar vector ξ is orthogonal to the fluid flow and of the infinite kind, we have in appropriate coordinates

$$\xi = r\frac{\partial}{\partial r}, \tag{3.63}$$

$$ds^2 = -r^2dt^2 + \frac{e^{2\Psi(t)}}{r^2}dr^2 + S(t)^2d\Omega^2, \tag{3.64}$$

where the self-similar variable is given by $\xi = t$.

Not as homothetic solutions in the titled case, kinematic self-similar solutions have a characteristic structure. We now show an example of them in the case of self-similarity of the second kind, where a kinematic self-similar vector is titled to the fluid flow. In this case, the Einstein equations imply that the quantities m, μ and p must be of the following form:

$$\frac{2Gm}{r} = M_1(\xi) + \frac{r^2}{t^2} M_2(\xi), \tag{3.65}$$

$$8\pi G\mu r^2 = W_1(\xi) + \frac{r^2}{t^2} W_2(\xi), \tag{3.66}$$

$$8\pi G p r^2 = P_1(\xi) + \frac{r^2}{t^2} P_2(\xi), \tag{3.67}$$

where $\xi = r/(\alpha t)^{1/\alpha}$. In other words, dimensionless quantities on the left hand side are decomposed into two parts, one remains constant and the other behaves as $(r/t)^2 \propto r^{2(1-\alpha)}$as ξ is fixed. Then, the original partial differential equations are satisfied for each of the $O(1)$ and $O[(r/t)^2]$ terms. The eqns (3.28)-(3.35) for a perfect fluid then reduce to the following:

$$M_1 + M_1' = W_1 S^2 (S + S'), \tag{3.68}$$

$$3M_2 + M_2' = W_2 S^2 (S + S'), \tag{3.69}$$

$$M_1' = -P_1 S^2 S', \tag{3.70}$$

$$2\alpha M_2 + M_2' = -P_2 S^2 S', \tag{3.71}$$

$$M_1 = S[1 - e^{-2\Psi}(S + S')^2], \tag{3.72}$$

$$\alpha^2 M_2 = SS'^2 e^{-2\Phi}, \tag{3.73}$$

$$(P_1 + W_1)\Phi' = 2P_1 - P_1', \tag{3.74}$$

$$(P_2 + W_2)\Phi' = -P_2', \tag{3.75}$$

$$W_1' S = -(P_1 + W_1)(\Psi' S + 2S'), \tag{3.76}$$

$$(2\alpha W_2 + W_2')S = -(P_2 + W_2)(\Psi' S + 2S'), \tag{3.77}$$

$$S'' + S' = S'\Phi' + (S + S')\Psi', \tag{3.78}$$

$$S'(S' + 2\Psi' S) = \alpha^2 W_2 S^2 e^{2\Phi}, \tag{3.79}$$

$$2S(S'' + 2S') - 2\Psi' S(S + S') = -S^2 - S^2 + e^{2\Psi}(1 - W_1 S^2), \tag{3.80}$$

$$2S(S'' + \alpha S' - \Phi' S') + S'^2 = -\alpha^2 P_2 S^2 e^{2\Phi}, \tag{3.81}$$

$$(S + S')(S + S' + 2\Phi' S) = (1 + P_1 S^2)e^{2\Psi}, \tag{3.82}$$

where we have omitted the bars of $\bar{\Phi}$ and $\bar{\psi}$ in (3.55) for simplicity and the prime denotes the derivative with respect to $ln\xi$. A similar structure of basic equations can be found for kinematic self-similar solutions of the second, zeroth and infinite kinds both in the titles and orthogonal cases and of the second and zeroth kind in the parallel case. The exceptions are the first kind in the titled, parallel and orthogonal cases and the infinite kind in the parallel case.

It is interesting to consider the spherically symmetric self-similar solutions of the infinite kind with a kinematic self-similar vector parallel to the fluid flow. The metric form demanded by this self-similarity, which is given by eqn(3.61), is nothing but the general form of the line element in spherically symmetric static spacetimes when the chosen radial coordinate is the radial physical length. Therefore, all static solutions have a kinematic self-similar vector of the infinite kind that is parallel to the fluid flow. Inversely, all spherically symmetric solutions with a kinematic self-similar vector of the infinite kind parallel to the fluid flow are static. The equation of state is not restricted at all.

```
c = {t,r,θ,ϕ}

{t, r, θ, ϕ}

gmat =
{{-(Exp[2ϕ[t,r]]),0,0,0},{0,Exp[2φ[t,r]],0,0},{0,0,R[t,r]^2,0},{0,0,0,R[t,r]^2Sin[θ]^2
}}

{{-e^(2ϕ[t,r]), 0, 0, 0}, {0, e^(2φ[t,r]), 0, 0}, {0, 0, R[t, r]^2, 0}, {0, 0, 0, R[t, r]^2 Sin[θ]^2}}

g = {-(Exp[2ϕ[t,r]]),0,0,0,0,Exp[2φ[t,r]],0,0,0,0,R[t,r]^2,0,0,0,0,R[t,r]^2Sin[θ]^2}

{-e^(2ϕ[t,r]), 0, 0, 0, 0, e^(2φ[t,r]), 0, 0, 0, 0, R[t, r]^2, 0, 0, 0, 0, R[t, r]^2 Sin[θ]^2}

<<tensorpak.m

gupper[c, g]
```

$$\left\{\left\{-e^{-2\phi[t,r]}, 0, 0, 0\right\}, \left\{0, e^{-2\varphi[t,r]}, 0, 0\right\}, \left\{0, 0, \frac{1}{R[t, r]^2}, 0\right\}, \left\{0, 0, 0, \frac{Csc[\theta]^2}{R[t, r]^2}\right\}\right\}$$

ChristoffelSymbol2[c, g]

{11, 1} = $\phi^{(1,0)}[t, r]$

{11, 2} = $e^{2\phi[t,r]-2\varphi[t,r]}\,\phi^{(0,1)}[t, r]$

{12, 1} = $\phi^{(0,1)}[t, r]$

{12, 2} = $\varphi^{(1,0)}[t, r]$

{13, 3} = $\dfrac{R^{(1,0)}[t, r]}{R[t, r]}$

{14, 4} = $\dfrac{R^{(1,0)}[t, r]}{R[t, r]}$

{21, 1} = $\phi^{(0,1)}[t, r]$

{21, 2} = $\varphi^{(1,0)}[t, r]$

{22, 1} = $e^{-2\phi[t,r]+2\varphi[t,r]}\,\varphi^{(1,0)}[t, r]$

{22, 2} = $\varphi^{(0,1)}[t, r]$

{23, 3} = $\dfrac{R^{(0,1)}[t, r]}{R[t, r]}$

{24, 4} = $\dfrac{R^{(0,1)}[t, r]}{R[t, r]}$

{31, 3} = $\dfrac{R^{(1,0)}[t, r]}{R[t, r]}$

{32, 3} = $\dfrac{R^{(0,1)}[t, r]}{R[t, r]}$

{33, 1} = $e^{-2\phi[t,r]}\,R[t, r]\,R^{(1,0)}[t, r]$

{33, 2} = $-e^{-2\varphi[t,r]}\,R[t, r]\,R^{(0,1)}[t, r]$

{34, 4} = $\mathrm{Cot}[\theta]$

{41, 4} = $\dfrac{R^{(1,0)}[t, r]}{R[t, r]}$

{42, 4} = $\dfrac{R^{(0,1)}[t, r]}{R[t, r]}$

{43, 4} = $\mathrm{Cot}[\theta]$

{44, 1} = $e^{-2\phi[t,r]}\,R[t, r]\,\mathrm{Sin}[\theta]^2\,R^{(1,0)}[t, r]$

{44, 2} = $-e^{-2\varphi[t,r]}\,R[t, r]\,\mathrm{Sin}[\theta]^2\,R^{(0,1)}[t, r]$

{44, 3} = $-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

Riemann[c, g]

Riemann [11,14] = $e^{-2\phi[t,r]}$ Derivative[1, 0]'[ϕ][t, r]

Riemann [11,24] = $e^{-2\phi[t,r]}$ Derivative[0, 1]'[ϕ][t, r]

Riemann [11,41] = $-e^{-2\phi[t,r]}$ Derivative[1, 0]'[ϕ][t, r]

Riemann [11,42] = $-e^{-2\phi[t,r]}$ Derivative[0, 1]'[ϕ][t, r]

$$\text{Riemann}[12,12] = e^{-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]\,(-\phi^{(0,1)}[t,r]+\varphi^{(0,1)}[t,r]) - \phi^{(0,2)}[t,r] + e^{-2\phi[t,r]+2\varphi[t,r]}\left(-\phi^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]+\varphi^{(1,0)}[t,r]^2+\varphi^{(2,0)}[t,r]\right)\right)$$

$$\text{Riemann}[12,14] = -e^{-2\varphi[t,r]}\,\text{Derivative}[0,1]'[\phi][t,r]$$

$$\text{Riemann}[12,21] = e^{-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]^2-\phi^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+\phi^{(0,2)}[t,r]\right) + e^{-2\phi[t,r]}\left((\phi^{(1,0)}[t,r]-\varphi^{(1,0)}[t,r])\,\varphi^{(1,0)}[t,r]-\varphi^{(2,0)}[t,r]\right)$$

$$\text{Riemann}[12,41] = e^{-2\varphi[t,r]}\,\text{Derivative}[0,1]'[\phi][t,r]$$

$$\text{Riemann}[13,13] = \frac{-e^{-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+e^{-2\phi[t,r]}\left(-R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]+R^{(2,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[13,23] = \frac{e^{-2\phi[t,r]}\left(-\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]-R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]+R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[13,31] = \frac{e^{-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+e^{-2\phi[t,r]}\left(R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]-R^{(2,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[13,32] = \frac{e^{-2\phi[t,r]}\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[14,14] = \frac{-e^{-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+e^{-2\phi[t,r]}\left(-R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]+R^{(2,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[14,24] = \frac{e^{-2\phi[t,r]}\left(-\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]-R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]+R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[14,41] = \frac{e^{-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+e^{-2\phi[t,r]}\left(R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]-R^{(2,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[14,42] = \frac{e^{-2\phi[t,r]}\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[21,12] = e^{-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]^2-\phi^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+\phi^{(0,2)}[t,r]\right) + e^{-2\phi[t,r]}\left((\phi^{(1,0)}[t,r]-\varphi^{(1,0)}[t,r])\,\varphi^{(1,0)}[t,r]-\varphi^{(2,0)}[t,r]\right)$$

$$\text{Riemann}[21,14] = e^{-2\varphi[t,r]}\,\text{Derivative}[0,1]'[\phi][t,r]$$

$$\text{Riemann}[21,21] = -e^{-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]^2-\phi^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+\phi^{(0,2)}[t,r]\right) + e^{-2\phi[t,r]}\left(-\phi^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]+\varphi^{(1,0)}[t,r]^2+\varphi^{(2,0)}[t,r]\right)$$

$$\text{Riemann}[21,41] = -e^{-2\varphi[t,r]}\,\text{Derivative}[0,1]'[\phi][t,r]$$

$$\text{Riemann}[23,13] = \frac{e^{-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[23,23] = \frac{e^{-2\varphi[t,r]}\left(R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]-R^{(0,2)}[t,r]\right)+e^{-2\phi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]}{R[t,r]}$$

$$\text{Riemann}[23,31] = \frac{e^{-2\varphi[t,r]}\left(-\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]-R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]+R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[23,32] = \frac{e^{-2\varphi[t,r]}\left(-R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+R^{(0,2)}[t,r]\right)-e^{-2\phi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]}{R[t,r]}$$

$$\text{Riemann}[24,14] = \frac{e^{-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[24,24] = \frac{e^{-2\varphi[t,r]}\left(R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]-R^{(0,2)}[t,r]\right)+e^{-2\phi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]}{R[t,r]}$$

$$\text{Riemann}[24,41] = \frac{e^{-2\varphi[t,r]}\left(-\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]-R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]+R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[24,42] = \frac{e^{-2\varphi[t,r]}\left(-R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+R^{(0,2)}[t,r]\right)-e^{-2\phi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]}{R[t,r]}$$

$$\text{Riemann}[31,13]=\frac{e^{-2\phi[t,r]}\left(e^{2\phi[t,r]-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]-R^{(2,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[31,23]=\frac{e^{-2\phi[t,r]}\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[31,31]=-\frac{e^{-2\phi[t,r]}\left(e^{2\phi[t,r]-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]-R^{(2,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[31,32]=\frac{e^{-2\phi[t,r]}\left(-\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]-R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]+R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[32,13]=\frac{e^{-2\varphi[t,r]}\left(-\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]-R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]+R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[32,23]=\frac{e^{-2\varphi[t,r]}\left(-R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+R^{(0,2)}[t,r]-e^{-2\phi[t,r]+2\varphi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[32,31]=\frac{e^{-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[32,32]=\frac{e^{-2\varphi[t,r]}\left(R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]-R^{(0,2)}[t,r]+e^{-2\phi[t,r]+2\varphi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[34,34]=\frac{1-e^{-2\varphi[t,r]}R^{(0,1)}[t,r]^2+e^{-2\phi[t,r]}R^{(1,0)}[t,r]^2}{R[t,r]^2}$$

$$\text{Riemann}[34,43]=\frac{-1+e^{-2\varphi[t,r]}R^{(0,1)}[t,r]^2-e^{-2\phi[t,r]}R^{(1,0)}[t,r]^2}{R[t,r]^2}$$

$$\text{Riemann}[41,14]=\frac{e^{-2\phi[t,r]}\left(e^{2\phi[t,r]-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]-R^{(2,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[41,24]=\frac{e^{-2\phi[t,r]}\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[41,41]=-\frac{e^{-2\phi[t,r]}\left(e^{2\phi[t,r]-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]-R^{(2,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[41,42]=\frac{e^{-2\phi[t,r]}\left(-\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]-R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]+R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[42,14]=\frac{e^{-2\varphi[t,r]}\left(-\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]-R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]+R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[42,24]=\frac{e^{-2\varphi[t,r]}\left(-R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+R^{(0,2)}[t,r]-e^{-2\phi[t,r]+2\varphi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[42,41]=\frac{e^{-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[42,42]=\frac{e^{-2\varphi[t,r]}\left(R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]-R^{(0,2)}[t,r]+e^{-2\phi[t,r]+2\varphi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]\right)}{R[t,r]}$$

$$\text{Riemann}[43,34]=\frac{-1+e^{-2\varphi[t,r]}R^{(0,1)}[t,r]^2-e^{-2\phi[t,r]}R^{(1,0)}[t,r]^2}{R[t,r]^2}$$

$$\text{Riemann}[43,43]=\frac{1-e^{-2\varphi[t,r]}R^{(0,1)}[t,r]^2+e^{-2\phi[t,r]}R^{(1,0)}[t,r]^2}{R[t,r]^2}$$

RicciTensor[c, g]

$$\text{Ricci Tensor [11]} = -e^{2\phi[t,r]-2\varphi[t,r]}\left(\phi^{(0,1)}[t,r]^2-\phi^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+\phi^{(0,2)}[t,r]\right)-\phi^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]+\varphi^{(1,0)}[t,r]^2-\frac{2\left(e^{2\phi[t,r]-2\varphi[t,r]}R^{(0,1)}[t,r]\,\phi^{(0,1)}[t,r]+R^{(1,0)}[t,r]\,\phi^{(1,0)}[t,r]-R^{(2,0)}[t,r]\right)}{R[t,r]}+\varphi^{(2,0)}[t,r]$$

$$\text{Ricci Tensor [12]} = -\frac{2\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Ricci Tensor [13]} = 0$$

$$\text{Ricci Tensor [14]} = \text{Derivative}[1,0]'[\phi][t,r]$$

$$\text{Ricci Tensor [21]} = -\frac{2\left(\phi^{(0,1)}[t,r]\,R^{(1,0)}[t,r]+R^{(0,1)}[t,r]\,\varphi^{(1,0)}[t,r]-R^{(1,1)}[t,r]\right)}{R[t,r]}$$

$$\text{Ricci Tensor [22]} = \phi^{(0,1)}[t,r]^2-\phi^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+\phi^{(0,2)}[t,r]+\frac{2\left(-R^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+R^{(0,2)}[t,r]-e^{-2\phi[t,r]+2\varphi[t,r]}R^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]\right)}{R[t,r]}+e^{-2\phi[t,r]+2\varphi[t,r]}\left(\left(\phi^{(1,0)}[t,r]-\varphi^{(1,0)}[t,r]\right)\varphi^{(1,0)}[t,r]-\varphi^{(2,0)}[t,r]\right)$$

$$\text{Ricci Tensor [23]} = 0$$

$$\text{Ricci Tensor [24]} = \text{Derivative}[0,1]'[\phi][t,r]$$

$$\text{Ricci Tensor [31]} = 0$$

$$\text{Ricci Tensor [32]} = 0$$

$$\text{Ricci Tensor [33]} = e^{-2(\phi[t,r]+\varphi[t,r])}\left(e^{2\phi[t,r]}\left(R^{(0,1)}[t,r]^2+R[t,r]\,R^{(0,1)}[t,r]\left(\phi^{(0,1)}[t,r]-\varphi^{(0,1)}[t,r]\right)+R[t,r]\,R^{(0,2)}[t,r]\right)-e^{2\varphi[t,r]}\left(e^{2\phi[t,r]}+R^{(1,0)}[t,r]^2+R[t,r]\,R^{(1,0)}[t,r]\left(-\phi^{(1,0)}[t,r]+\varphi^{(1,0)}[t,r]\right)+R[t,r]\,R^{(2,0)}[t,r]\right)\right)$$

$$\text{Ricci Tensor [34]} = 0$$

$$\text{Ricci Tensor [41]} = 0$$

$$\text{Ricci Tensor [42]} = 0$$

$$\text{Ricci Tensor [43]} = 0$$

$$\text{Ricci Tensor [44]} = e^{-2(\phi[t,r]+\varphi[t,r])}\,\text{Sin}[\theta]^2\left(e^{2\phi[t,r]}\left(R^{(0,1)}[t,r]^2+R[t,r]\,R^{(0,1)}[t,r]\left(\phi^{(0,1)}[t,r]-\varphi^{(0,1)}[t,r]\right)+R[t,r]\,R^{(0,2)}[t,r]\right)-e^{2\varphi[t,r]}\left(e^{2\phi[t,r]}+R^{(1,0)}[t,r]^2+R[t,r]\,R^{(1,0)}[t,r]\left(-\phi^{(1,0)}[t,r]+\varphi^{(1,0)}[t,r]\right)+R[t,r]\,R^{(2,0)}[t,r]\right)\right)$$

ScalarCurvature[c, g]

$$\frac{1}{R[t,r]^2}\left(e^{-2(\phi[t,r]+\varphi[t,r])}\left(2\,e^{2\phi[t,r]}\left(R^{(0,1)}[t,r]^2+2\,R[t,r]\,R^{(0,1)}[t,r]\left(\phi^{(0,1)}[t,r]-\varphi^{(0,1)}[t,r]\right)+R[t,r]\left(2\,R^{(0,2)}[t,r]+R[t,r]\left(\phi^{(0,1)}[t,r]^2-\phi^{(0,1)}[t,r]\,\varphi^{(0,1)}[t,r]+\phi^{(0,2)}[t,r]\right)\right)\right)-2\,e^{2\varphi[t,r]}\left(e^{2\phi[t,r]}+R^{(1,0)}[t,r]^2+2\,R[t,r]\,R^{(1,0)}[t,r]\left(-\phi^{(1,0)}[t,r]+\varphi^{(1,0)}[t,r]\right)+2\,R[t,r]\,R^{(2,0)}[t,r]+R[t,r]^2\left(-\phi^{(1,0)}[t,r]\,\varphi^{(1,0)}[t,r]+\varphi^{(1,0)}[t,r]^2+\varphi^{(2,0)}[t,r]\right)\right)\right)\right)$$

KritchmanTensor[c, g]

KritchmanTensor [12,12] = $\frac{2 e^{-2\psi} (-3 a r + t)^2 \text{Csc}[\theta]^2}{9 b^2 r^3 (a r - t)^3 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [12,13] = $\frac{2 e^{-2\psi} (3 a r - t) \text{Cot}[\theta] \text{Csc}[\theta]^2}{3 b^2 r^2 (-a r + t)^2 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [12,14] = $-\frac{2 e^{-2(\phi+\psi)} (3 a r - t) \left(2 b^2 + 9 e^{2\phi} (a - \frac{t}{r})^{2/3} \text{Csc}[\theta]^2\right)}{27 b^2 r (a r - t)^3}$

KritchmanTensor [12,21] = $-\frac{2 e^{-2\psi} (-3 a r + t)^2 \text{Csc}[\theta]^2}{9 b^2 r^3 (a r - t)^3 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [12,24] = $\frac{4 e^{-2(\phi+\psi)} (3 a r - t) t}{27 r^2 (a r - t)^3}$

KritchmanTensor [12,31] = $-\frac{2 e^{-2\psi} (3 a r - t) \text{Cot}[\theta] \text{Csc}[\theta]^2}{3 b^2 r^2 (-a r + t)^2 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [12,41] = $\frac{2 e^{-2(\phi+\psi)} (3 a r - t) \left(2 b^2 r (a - \frac{t}{r})^{1/3} + 9 e^{2\phi} (a r - t) \text{Csc}[\theta]^2\right)}{27 b^2 r^2 (a r - t)^3 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [12,42] = $-\frac{4 e^{-2(\phi+\psi)} (3 a r - t) t}{27 r^2 (a r - t)^3}$

KritchmanTensor [13,12] = $\frac{2 (3 a r - t) \text{Cot}[\theta] \text{Csc}[\theta]^2}{3 b^4 r^3 (a r - t)^3 (a - \frac{t}{r})^{2/3}}$

KritchmanTensor [13,13] =

$$\frac{8 e^{-4\phi}}{81 (-a r + t)^4} - \frac{8 e^{-2(\phi+\psi)} t^2}{81 r^2 (-a r + t)^4} - \frac{4 e^{-2\phi} (a - \frac{t}{r})^{2/3} \text{Csc}[\theta]^2}{9 b^2 (-a r + t)^4} + \frac{2 \text{Cot}[\theta]^2 \text{Csc}[\theta]^2}{b^4 r^4 (a - \frac{t}{r})^{8/3}}$$

KritchmanTensor [13,14] = $-\frac{2 e^{-2\phi} \text{Cot}[\theta] \left(4 b^2 + 9 e^{2\phi} (a - \frac{t}{r})^{2/3} \text{Csc}[\theta]^2\right)}{9 b^4 r (a r - t)^3 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [13,21] = $\frac{2 (-3 a r + t) \text{Cot}[\theta] \text{Csc}[\theta]^2}{3 b^4 r^3 (a r - t)^3 (a - \frac{t}{r})^{2/3}}$

KritchmanTensor [13,23] = $\frac{8 e^{-2(2\phi+\psi)} t (-e^{2\psi} r^2 + e^{2\phi} t^2)}{81 r^3 (-a r + t)^4}$

KritchmanTensor [13,24] = $\frac{4 e^{-2\phi} t \text{Cot}[\theta]}{9 b^2 r^3 (-a r + t)^2 (a - \frac{t}{r})^{4/3}}$

KritchmanTensor [13,31] =

$$\frac{e^{-2(2\phi+\psi)} \left(-8 b^4 (e^{2\psi} r^2 - e^{2\phi} t^2) + 3 e^{2(\phi+\psi)} r (a - \frac{t}{r})^{1/3} \left(4 b^2 r (a - \frac{t}{r})^{1/3} - 9 e^{2\phi} (a r - t) \text{Cot}[\theta]^2\right) \text{Csc}[\theta]^2\right)}{81 b^4 r^2 (-a r + t)^4}$$

KritchmanTensor [13,32] = $\frac{8 e^{-2(2\phi+\psi)} t (e^{2\psi} r^2 - e^{2\phi} t^2)}{81 r^3 (-a r + t)^4}$

KritchmanTensor [13,34] = $\frac{e^{-2(2\phi+\psi)} \left(-8 b^2 e^{2\psi} r^2 + 4 b^2 e^{2\phi} (-3 a r + t)^2 - 36 e^{2(\phi+\psi)} r^2 (a - \frac{t}{r})^{2/3}\right)}{27 b^2 r^2 (a r - t)^3}$

KritchmanTensor [13,41] = $\frac{2 e^{-2\phi} \text{Cot}[\theta] \left(4 b^2 r (a - \frac{t}{r})^{1/3} + 9 e^{2\phi} (a r - t) \text{Csc}[\theta]^2\right)}{9 b^4 r^2 (a r - t)^3 (a - \frac{t}{r})^{2/3}}$

KritchmanTensor [13,42] = $-\frac{4 e^{-2\phi} t \text{Cot}[\theta]}{9 b^2 r^3 (-a r + t)^2 (a - \frac{t}{r})^{4/3}}$

KritchmanTensor [13,43] = $\frac{e^{-2(2\phi+\psi)} \left(8 b^2 e^{2\psi} r^2 - 4 b^2 e^{2\phi} (-3 a r + t)^2 + 36 e^{2(\phi+\psi)} r^2 (a - \frac{t}{r})^{2/3}\right)}{27 b^2 r^2 (a r - t)^3}$

$$\text{KritchmanTensor}\ [14,12] = -\frac{2\,e^{-2\phi}\,(3\,a\,r-t)\,\text{Csc}[\theta]^2\,\left(2\,b^2+9\,e^{2\phi}\,(a-\frac{t}{r})^{2/3}\,\text{Csc}[\theta]^2\right)}{27\,b^4\,r^2\,(-a\,r+t)^4\,(a-\frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [14,13] = -\frac{2\,e^{-2\phi}\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2\,\left(4\,b^2+9\,e^{2\phi}\,(a-\frac{t}{r})^{2/3}\,\text{Csc}[\theta]^2\right)}{9\,b^4\,r\,(a\,r-t)^3\,(a-\frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [14,14] = \Big(2\,e^{-2\,(2\phi+\varphi)}\Big(4\,b^4\,r\left(a-\frac{t}{r}\right)^{2/3}(e^{2\varphi}\,r^2-e^{2\phi}\,t^2)+\frac{9}{2}\,e^{2\phi}\,(a\,r-t)\,\text{Csc}[\theta]^2\Big(2\,b^2\left(a-\frac{t}{r}\right)^{1/3}(4\,e^{2\varphi}\,r^2+e^{2\phi}\,(-3\,a\,r+t)^2)+9\,e^{2\,(\phi+\varphi)}\,r\,(a\,r-t)\,(3+\text{Cos}[2\,\theta])\,\text{Csc}[\theta]^2\Big)\Big)\Big)\Big/\left(81\,b^4\,r^3\,(-a\,r+t)^4\left(a-\frac{t}{r}\right)^{2/3}\right)$$

$$\text{KritchmanTensor}\ [14,21] = \frac{2\,e^{-2\phi}\,(3\,a\,r-t)\,\text{Csc}[\theta]^2\,\left(2\,b^2+9\,e^{2\phi}\,(a-\frac{t}{r})^{2/3}\,\text{Csc}[\theta]^2\right)}{27\,b^4\,r^2\,(-a\,r+t)^4\,(a-\frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [14,23] = \frac{4\,e^{-2\phi}\,t\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2}{9\,b^2\,r^3\,(-a\,r+t)^2\,(a-\frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [14,24] = \frac{4\,e^{-2\,(2\phi+\varphi)}\,t\,\left(-2\,b^2\,(e^{2\varphi}\,r^2-e^{2\phi}\,t^2)-9\,e^{2\,(\phi+\varphi)}\,r^2\,(a-\frac{t}{r})^{2/3}\,\text{Csc}[\theta]^2\right)}{81\,b^2\,r^3\,(-a\,r+t)^4}$$

$$\text{KritchmanTensor}\ [14,31] = \frac{2\,e^{-2\phi}\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2\,\left(4\,b^2+9\,e^{2\phi}\,(a-\frac{t}{r})^{2/3}\,\text{Csc}[\theta]^2\right)}{9\,b^4\,r\,(a\,r-t)^3\,(a-\frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [14,32] = -\frac{4\,e^{-2\phi}\,t\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2}{9\,b^2\,r^3\,(-a\,r+t)^2\,(a-\frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [14,34] = -\frac{4\,e^{-2\phi}\,(a-\frac{t}{r})^{2/3}\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2}{3\,b^2\,(a\,r-t)^3}$$

$$\text{KritchmanTensor}\ [14,41] = -\Big(2\,e^{-2\,(2\phi+\varphi)}\Big(4\,b^4\,r\left(a-\frac{t}{r}\right)^{2/3}(e^{2\varphi}\,r^2-e^{2\phi}\,t^2)+\frac{9}{2}\,e^{2\phi}\,(a\,r-t)\,\text{Csc}[\theta]^2\Big(2\,b^2\left(a-\frac{t}{r}\right)^{1/3}(4\,e^{2\varphi}\,r^2+e^{2\phi}\,(-3\,a\,r+t)^2)+9\,e^{2\,(\phi+\varphi)}\,r\,(a\,r-t)\,(3+\text{Cos}[2\,\theta])\,\text{Csc}[\theta]^2\Big)\Big)\Big)\Big/\left(81\,b^4\,r^3\,(-a\,r+t)^4\left(a-\frac{t}{r}\right)^{2/3}\right)$$

$$\text{KritchmanTensor}\ [14,42] = \frac{4\,e^{-2\,(2\phi+\varphi)}\,t\,\left(2\,b^2\,(e^{2\varphi}\,r^2-e^{2\phi}\,t^2)+9\,e^{2\,(\phi+\varphi)}\,r^2\,(a-\frac{t}{r})^{2/3}\,\text{Csc}[\theta]^2\right)}{81\,b^2\,r^3\,(-a\,r+t)^4}$$

$$\text{KritchmanTensor}\ [14,43] = \frac{4\,e^{-2\phi}\,(a-\frac{t}{r})^{2/3}\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2}{3\,b^2\,(a\,r-t)^3}$$

$$\text{KritchmanTensor}\ [21,12] = -\frac{2\,e^{-2\varphi}\,(-3\,a\,r+t)^2\,\text{Csc}[\theta]^2}{9\,b^2\,r^3\,(a\,r-t)^3\,(a-\frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [21,13] = \frac{2\,e^{-2\varphi}\,(-3\,a\,r+t)\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2}{3\,b^2\,r^3\,(a\,r-t)\,(a-\frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [21,14] = \frac{2\,e^{-2\,(\phi+\varphi)}\,(3\,a\,r-t)\,\left(2\,b^2+9\,e^{2\phi}\,(a-\frac{t}{r})^{2/3}\,\text{Csc}[\theta]^2\right)}{27\,b^2\,r\,(a\,r-t)^3}$$

$$\text{KritchmanTensor}\ [21,21] = \frac{2\,e^{-2\varphi}\,(-3\,a\,r+t)^2\,\text{Csc}[\theta]^2}{9\,b^2\,r^3\,(a\,r-t)^3\,(a-\frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [21,24] = -\frac{4\,e^{-2\,(\phi+\varphi)}\,(3\,a\,r-t)\,t}{27\,r^2\,(a\,r-t)^3}$$

$$\text{KritchmanTensor}\ [21,31] = -\frac{2\,e^{-2\varphi}\,(-3\,a\,r+t)\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2}{3\,b^2\,r^3\,(a\,r-t)\,(a-\frac{t}{r})^{4/3}}$$

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KritchmanTensor [21,41] $= -\frac{2 e^{-2(\phi+\psi)} (3 a r - t) \left(2 b^2 r (a - \frac{t}{r})^{1/3} + 9 e^{2\phi} (a r - t) \mathrm{Csc}[\theta]^2\right)}{27 b^2 r^2 (a r - t)^3 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [21,42] $= \frac{4 e^{-2(\phi+\psi)} (3 a r - t) t}{27 r^2 (a r - t)^3}$

KritchmanTensor [23,13] $= \frac{8 e^{-2(\phi+2\psi)} t (e^{2\psi} r^2 - e^{2\phi} t^2)}{81 r^3 (-a r + t)^4}$

KritchmanTensor [23,14] $= -\frac{4 e^{-2\psi} t \mathrm{Cot}[\theta]}{9 b^2 r^3 (-a r + t)^2 (a - \frac{t}{r})^{4/3}}$

KritchmanTensor [23,23] $= \frac{e^{-2(\phi+2\psi)} (-8 e^{2\psi} r^2 t^2 + 8 e^{2\phi} t^4)}{81 r^4 (-a r + t)^4}$

KritchmanTensor [23,31] $= \frac{e^{-2(\phi+2\psi)} (-8 e^{2\psi} r^2 t + 8 e^{2\phi} t^3)}{81 r^3 (-a r + t)^4}$

KritchmanTensor [23,32] $= \frac{8 e^{-2(\phi+2\psi)} t^2 (e^{2\psi} r^2 - e^{2\phi} t^2)}{81 r^4 (-a r + t)^4}$

KritchmanTensor [23,34] $= \frac{8 e^{-2(\phi+\psi)} t}{27 r (a r - t)^3}$

KritchmanTensor [23,41] $= \frac{4 e^{-2\psi} t \mathrm{Cot}[\theta]}{9 b^2 r^3 (-a r + t)^2 (a - \frac{t}{r})^{4/3}}$

KritchmanTensor [23,43] $= -\frac{8 e^{-2(\phi+\psi)} t}{27 r (a r - t)^3}$

KritchmanTensor [24,12] $= -\frac{4 e^{-2\psi} (3 a r - t) t \mathrm{Csc}[\theta]^2}{27 b^2 r^3 (-a r + t)^4 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [24,13] $= -\frac{4 e^{-2\psi} t \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2}{9 b^2 r^3 (-a r + t)^2 (a - \frac{t}{r})^{4/3}}$

KritchmanTensor [24,14] $= \frac{4 e^{-2(\phi+2\psi)} t \left(2 b^2 (e^{2\psi} r^2 - e^{2\phi} t^2) + 9 e^{2(\phi+\psi)} r^2 (a - \frac{t}{r})^{2/3} \mathrm{Csc}[\theta]^2\right)}{81 b^2 r^3 (-a r + t)^4}$

KritchmanTensor [24,21] $= \frac{4 e^{-2\psi} (3 a r - t) t \mathrm{Csc}[\theta]^2}{27 b^2 r^3 (-a r + t)^4 (a - \frac{t}{r})^{1/3}}$

KritchmanTensor [24,24] $= \frac{e^{-2(\phi+2\psi)} (-8 e^{2\psi} r^2 t^2 + 8 e^{2\phi} t^4)}{81 r^4 (-a r + t)^4}$

KritchmanTensor [24,31] $= \frac{4 e^{-2\psi} t \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2}{9 b^2 r^3 (-a r + t)^2 (a - \frac{t}{r})^{4/3}}$

KritchmanTensor [24,41] $= \frac{4 e^{-2(\phi+2\psi)} t \left(-2 b^2 (e^{2\psi} r^2 - e^{2\phi} t^2) - 9 e^{2(\phi+\psi)} r^2 (a - \frac{t}{r})^{2/3} \mathrm{Csc}[\theta]^2\right)}{81 b^2 r^3 (-a r + t)^4}$

KritchmanTensor [24,42] $= \frac{8 e^{-2(\phi+2\psi)} t^2 (e^{2\psi} r^2 - e^{2\phi} t^2)}{81 r^4 (-a r + t)^4}$

KritchmanTensor [31,12] $= \frac{2 (-3 a r + t) \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2}{3 b^4 r^3 (a r - t)^3 (a - \frac{t}{r})^{2/3}}$

KritchmanTensor [31,13] =

$$-\frac{8 e^{-4\phi}}{81 (-a r + t)^4} + \frac{8 e^{-2(\phi+\psi)} t^2}{81 r^2 (-a r + t)^4} + \frac{8 e^{-2\phi} (a - \frac{t}{r})^{2/3} \mathrm{Csc}[\theta]^2}{9 b^2 (-a r + t)^4} - \frac{2 \mathrm{Cot}[\theta]^2 \mathrm{Csc}[\theta]^2}{b^4 r^4 (a - \frac{t}{r})^{4/3}}$$

KritchmanTensor [31,14] $= \frac{2 e^{-2\phi} \mathrm{Cot}[\theta] \left(4 b^2 + 9 e^{2\phi} (a - \frac{t}{r})^{2/3} \mathrm{Csc}[\theta]^2\right)}{9 b^4 r (a r - t)^3 (a - \frac{t}{r})^{1/3}}$

$$\text{KritchmanTensor}\ [31,21] = \frac{2\,(3\,a\,r - t)\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2}{3\,b^4\,r^3\,(a\,r - t)^3\,(a - \frac{t}{r})^{2/3}}$$

$$\text{KritchmanTensor}\ [31,23] = \frac{8\,e^{-2\,(2\,\phi+\varphi)}\,t\,(e^{2\,\varphi}\,r^2 - e^{2\,\phi}\,t^2)}{81\,r^3\,(-a\,r + t)^4}$$

$$\text{KritchmanTensor}\ [31,24] = -\frac{4\,e^{-2\,\phi}\,t\,\text{Cot}[\theta]}{9\,b^2\,r^3\,(-a\,r + t)^2\,(a - \frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [31,31] = \left(2\,e^{-2\,(2\,\phi+\varphi)}\left(4\,b^4\left(a - \frac{t}{r}\right)^{2/3}(e^{2\,\varphi}\,r^2 - e^{2\,\phi}\,t^2) + 9\,e^{2\,(\phi+\varphi)}\,(a\,r - t)\left(-4\,b^2\,r\left(a - \frac{t}{r}\right)^{1/3} + 9\,e^{2\,\phi}\,(a\,r - t)\,\text{Cot}[\theta]^2\right)\text{Csc}[\theta]^2\right)\right)\Big/\left(81\,b^4\,r^2\,(-a\,r + t)^4\left(a - \frac{t}{r}\right)^{2/3}\right)$$

$$\text{KritchmanTensor}\ [31,32] = \frac{8\,e^{-2\,(2\,\phi+\varphi)}\,t\,(-e^{2\,\varphi}\,r^2 + e^{2\,\phi}\,t^2)}{81\,r^3\,(-a\,r + t)^4}$$

$$\text{KritchmanTensor}\ [31,34] = \frac{e^{-2\,(2\,\phi+\varphi)}\left(8\,b^2\,e^{2\,\varphi}\,r^2 - 4\,b^2\,e^{2\,\phi}\,(-3\,a\,r + t)^2 + 36\,e^{2\,(\phi+\varphi)}\,r^2\,(a - \frac{t}{r})^{2/3}\right)}{27\,b^2\,r^2\,(a\,r - t)^3}$$

$$\text{KritchmanTensor}\ [31,41] = -\frac{2\,e^{-2\,\phi}\,\text{Cot}[\theta]\left(4\,b^2\,r\,(a - \frac{t}{r})^{1/3} + 9\,e^{2\,\phi}\,(a\,r - t)\,\text{Csc}[\theta]^2\right)}{9\,b^4\,r^2\,(a\,r - t)^3\,(a - \frac{t}{r})^{2/3}}$$

$$\text{KritchmanTensor}\ [31,42] = \frac{4\,e^{-2\,\phi}\,t\,\text{Cot}[\theta]}{9\,b^2\,r^3\,(-a\,r + t)^2\,(a - \frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [31,43] = \frac{e^{-2\,(2\,\phi+\varphi)}\left(-8\,b^2\,e^{2\,\varphi}\,r^2 + 4\,b^2\,e^{2\,\phi}\,(-3\,a\,r + t)^2 - 36\,e^{2\,(\phi+\varphi)}\,r^2\,(a - \frac{t}{r})^{2/3}\right)}{27\,b^2\,r^2\,(a\,r - t)^3}$$

$$\text{KritchmanTensor}\ [32,13] = \frac{e^{-2\,(\phi+2\,\varphi)}\,(-8\,e^{2\,\varphi}\,r^2\,t + 8\,e^{2\,\phi}\,t^3)}{81\,r^3\,(-a\,r + t)^4}$$

$$\text{KritchmanTensor}\ [32,14] = \frac{4\,e^{-2\,\varphi}\,t\,\text{Cot}[\theta]}{9\,b^2\,r^3\,(-a\,r + t)^2\,(a - \frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [32,23] = \frac{8\,e^{-2\,(\phi+2\,\varphi)}\,t^2\,(e^{2\,\varphi}\,r^2 - e^{2\,\phi}\,t^2)}{81\,r^4\,(-a\,r + t)^4}$$

$$\text{KritchmanTensor}\ [32,31] = \frac{8\,e^{-2\,(\phi+2\,\varphi)}\,t\,(e^{2\,\varphi}\,r^2 - e^{2\,\phi}\,t^2)}{81\,r^3\,(-a\,r + t)^4}$$

$$\text{KritchmanTensor}\ [32,32] = \frac{e^{-2\,(\phi+2\,\varphi)}\,(-8\,e^{2\,\varphi}\,r^2\,t^2 + 8\,e^{2\,\phi}\,t^4)}{81\,r^4\,(-a\,r + t)^4}$$

$$\text{KritchmanTensor}\ [32,34] = -\frac{8\,e^{-2\,(\phi+\varphi)}\,t}{27\,r\,(a\,r - t)^3}$$

$$\text{KritchmanTensor}\ [32,41] = -\frac{4\,e^{-2\,\varphi}\,t\,\text{Cot}[\theta]}{9\,b^2\,r^3\,(-a\,r + t)^2\,(a - \frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [32,43] = \frac{8\,e^{-2\,(\phi+\varphi)}\,t}{27\,r\,(a\,r - t)^3}$$

$$\text{KritchmanTensor}\ [34,13] = \frac{4\,e^{-2\,(\phi+\varphi)}\left(2\,b^2\,e^{2\,\varphi}\,r^2 - b^2\,e^{2\,\phi}\,(-3\,a\,r + t)^2 + 9\,e^{2\,(\phi+\varphi)}\,r^2\,(a - \frac{t}{r})^{2/3}\right)\text{Csc}[\theta]^2}{27\,b^4\,r^3\,(-a\,r + t)^4\,(a - \frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [34,14] = \frac{4\,\text{Cot}[\theta]\,\text{Csc}[\theta]^2}{3\,b^4\,r^2\,(a\,r - t)^3\,(a - \frac{t}{r})^{2/3}}$$

$$\text{KritchmanTensor}\ [34,23] = \frac{8\,e^{-2\,\phi}\,t\,\text{Csc}[\theta]^2}{27\,b^2\,r^2\,(-a\,r + t)^4\,(a - \frac{t}{r})^{1/3}}$$

KritchmanTensor [34,31] $= \frac{4 e^{-2(\phi+\psi)} \left(-2 b^2 e^{2\psi} r^2 + b^2 e^{2\phi} (-3 a r + t)^2 - 9 e^{2(\phi+\psi)} r^2 \left(a - \frac{t}{r}\right)^{2/3}\right) \mathrm{Csc}[\theta]^2}{27 b^4 r^3 (-a r + t)^4 \left(a - \frac{t}{r}\right)^{1/3}}$

KritchmanTensor [34,32] $= -\frac{8 e^{-2\phi} t\, \mathrm{Csc}[\theta]^2}{27 b^2 r^2 (-a r + t)^4 \left(a - \frac{t}{r}\right)^{1/3}}$

KritchmanTensor [34,34] =

$$\frac{2 e^{-4(\phi+\psi)} \left(4 b^2 e^{2\psi} r^2 - b^2 e^{2\phi} (-3 a r + t)^2 + 9 e^{2(\phi+\psi)} r^2 \left(a - \frac{t}{r}\right)^{2/3}\right)^2}{81 b^4 r^4 (-a r + t)^4} - \frac{8 e^{-2\phi} \left(a - \frac{t}{r}\right)^{2/3} \mathrm{Csc}[\theta]^2}{9 b^2 (-a r + t)^4}$$

KritchmanTensor [34,41] $= -\frac{4 \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2}{3 b^4 r^2 (a r - t)^3 \left(a - \frac{t}{r}\right)^{2/3}}$

KritchmanTensor [34,43] =

$$-\frac{2 e^{-4(\phi+\psi)} \left(4 b^2 e^{2\psi} r^2 - b^2 e^{2\phi} (-3 a r + t)^2 + 9 e^{2(\phi+\psi)} r^2 \left(a - \frac{t}{r}\right)^{2/3}\right)^2}{81 b^4 r^4 (-a r + t)^4} + \frac{8 e^{-2\phi} \left(a - \frac{t}{r}\right)^{2/3} \mathrm{Csc}[\theta]^2}{9 b^2 (-a r + t)^4}$$

KritchmanTensor [41,12] $= \frac{2 e^{-2\phi} (3 a r - t) \mathrm{Csc}[\theta]^2 \left(2 b^2 r \left(a - \frac{t}{r}\right)^{1/3} + 9 e^{2\phi} (a r - t) \mathrm{Csc}[\theta]^2\right)}{27 b^4 r^3 (-a r + t)^4 \left(a - \frac{t}{r}\right)^{2/3}}$

KritchmanTensor [41,13] $= \frac{2 e^{-2\phi} \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2 \left(4 b^2 r \left(a - \frac{t}{r}\right)^{1/3} + 9 e^{2\phi} (a r - t) \mathrm{Csc}[\theta]^2\right)}{9 b^4 r^2 (a r - t)^3 \left(a - \frac{t}{r}\right)^{2/3}}$

KritchmanTensor [41,14] =

$$-\frac{1}{81 b^4 r^3 (-a r + t)^4 \left(a - \frac{t}{r}\right)^{2/3}} \left(2 e^{-2(2\phi+\psi)} \left(4 b^4 r \left(a - \frac{t}{r}\right)^{2/3} (e^{2\psi} r^2 - e^{2\phi} t^2) + \frac{9}{2} e^{2\phi} (a r - t) \mathrm{Csc}[\theta]^2 \left(2 b^2 \left(a - \frac{t}{r}\right)^{1/3} (4 e^{2\psi} r^2 + e^{2\phi} (-3 a r + t)^2) + 9 e^{2(\phi+\psi)} r (a r - t) (3 + \mathrm{Cos}[2\theta]) \mathrm{Csc}[\theta]^2\right)\right)\right)$$

KritchmanTensor [41,21] $= -\frac{2 e^{-2\phi} (3 a r - t) \mathrm{Csc}[\theta]^2 \left(2 b^2 r \left(a - \frac{t}{r}\right)^{1/3} + 9 e^{2\phi} (a r - t) \mathrm{Csc}[\theta]^2\right)}{27 b^4 r^3 (-a r + t)^4 \left(a - \frac{t}{r}\right)^{2/3}}$

KritchmanTensor [41,23] $= -\frac{4 e^{-2\phi} t\, \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2}{9 b^2 r^3 (-a r + t)^2 \left(a - \frac{t}{r}\right)^{4/3}}$

KritchmanTensor [41,24] $= \frac{4 e^{-2(2\phi+\psi)} t \left(2 b^2 (e^{2\psi} r^2 - e^{2\phi} t^2) + 9 e^{2(\phi+\psi)} r^2 \left(a - \frac{t}{r}\right)^{2/3} \mathrm{Csc}[\theta]^2\right)}{81 b^2 r^3 (-a r + t)^4}$

KritchmanTensor [41,31] $= -\frac{2 e^{-2\phi} \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2 \left(4 b^2 r \left(a - \frac{t}{r}\right)^{1/3} + 9 e^{2\phi} (a r - t) \mathrm{Csc}[\theta]^2\right)}{9 b^4 r^2 (a r - t)^3 \left(a - \frac{t}{r}\right)^{2/3}}$

KritchmanTensor [41,32] $= \frac{4 e^{-2\phi} t\, \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2}{9 b^2 r^3 (-a r + t)^2 \left(a - \frac{t}{r}\right)^{4/3}}$

KritchmanTensor [41,34] $= \frac{4 e^{-2\phi} \left(a - \frac{t}{r}\right)^{2/3} \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2}{3 b^2 (a r - t)^3}$

KritchmanTensor [41,41] =

$$\frac{1}{81 b^4 r^3 (-a r + t)^4 \left(a - \frac{t}{r}\right)^{2/3}} \left(2 e^{-2(2\phi+\psi)} \left(4 b^4 r \left(a - \frac{t}{r}\right)^{2/3} (e^{2\psi} r^2 - e^{2\phi} t^2) + \frac{9}{2} e^{2\phi} (a r - t) \mathrm{Csc}[\theta]^2 \left(2 b^2 \left(a - \frac{t}{r}\right)^{1/3} (4 e^{2\psi} r^2 + e^{2\phi} (-3 a r + t)^2) + 9 e^{2(\phi+\psi)} r (a r - t) (3 + \mathrm{Cos}[2\theta]) \mathrm{Csc}[\theta]^2\right)\right)\right)$$

KritchmanTensor [41,42] $= \frac{4 e^{-2(2\phi+\psi)} t \left(-2 b^2 (e^{2\psi} r^2 - e^{2\phi} t^2) - 9 e^{2(\phi+\psi)} r^2 \left(a - \frac{t}{r}\right)^{2/3} \mathrm{Csc}[\theta]^2\right)}{81 b^2 r^3 (-a r + t)^4}$

KritchmanTensor [41,43] $= -\frac{4 e^{-2\phi} \left(a - \frac{t}{r}\right)^{2/3} \mathrm{Cot}[\theta] \mathrm{Csc}[\theta]^2}{3 b^2 (a r - t)^3}$

$$\text{KritchmanTensor}\ [42,12] = \frac{4\,e^{-2\varphi}\,(3\,a\,r-t)\,t\,Csc[\theta]^2}{27\,b^2\,r^3\,(-a\,r+t)^4\,(a-\frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [42,13] = \frac{4\,e^{-2\varphi}\,t\,Cot[\theta]\,Csc[\theta]^2}{9\,b^2\,r^3\,(-a\,r+t)^2\,(a-\frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [42,14] = \frac{4\,e^{-2\,(\phi+2\,\varphi)}\,t\,\left(-2\,b^2\,(e^{2\,\varphi}\,r^2-e^{2\,\phi}\,t^2)-9\,e^{2\,(\phi+\varphi)}\,r^2\,(a-\frac{t}{r})^{2/3}\,Csc[\theta]^2\right)}{81\,b^2\,r^3\,(-a\,r+t)^4}$$

$$\text{KritchmanTensor}\ [42,21] = -\frac{4\,e^{-2\varphi}\,(3\,a\,r-t)\,t\,Csc[\theta]^2}{27\,b^2\,r^3\,(-a\,r+t)^4\,(a-\frac{t}{r})^{1/3}}$$

$$\text{KritchmanTensor}\ [42,24] = \frac{8\,e^{-2\,(\phi+2\,\varphi)}\,t^2\,(e^{2\,\varphi}\,r^2-e^{2\,\phi}\,t^2)}{81\,r^4\,(-a\,r+t)^4}$$

$$\text{KritchmanTensor}\ [42,31] = -\frac{4\,e^{-2\varphi}\,t\,Cot[\theta]\,Csc[\theta]^2}{9\,b^2\,r^3\,(-a\,r+t)^2\,(a-\frac{t}{r})^{4/3}}$$

$$\text{KritchmanTensor}\ [42,41] = \frac{4\,e^{-2\,(\phi+2\,\varphi)}\,t\,\left(2\,b^2\,(e^{2\,\varphi}\,r^2-e^{2\,\phi}\,t^2)+9\,e^{2\,(\phi+\varphi)}\,r^2\,(a-\frac{t}{r})^{2/3}\,Csc[\theta]^2\right)}{81\,b^2\,r^3\,(-a\,r+t)^4}$$

$$\text{KritchmanTensor}\ [42,42] = \frac{e^{-2\,(\phi+2\,\varphi)}\,(-8\,e^{2\,\varphi}\,r^2\,t^2+8\,e^{2\,\phi}\,t^4)}{81\,r^4\,(-a\,r+t)^4}$$

$$\text{KritchmanTensor}\ [43,13] = \frac{1}{27\,b^4\,r^2\,(a\,r-t)^5}$$
$$\left(4\,e^{-2\,(\phi+\varphi)}\left(a-\frac{t}{r}\right)^{1/3}\left(9\,e^{2\,(\phi+\varphi)}\,r\,t+9\,a^2\,b^2\,e^{2\,\phi}\,r^2\left(a-\frac{t}{r}\right)^{1/3}-b^2\left(a-\frac{t}{r}\right)^{1/3}(2\,e^{2\,\varphi}\,r^2-e^{2\,\phi}\,t^2)-3\,a\,e^{2\,\phi}\,r\left(3\,e^{2\,\varphi}\,r+2\,b^2\,t\left(a-\frac{t}{r}\right)^{1/3}\right)\right)Csc[\theta]^2\right)$$

$$\text{KritchmanTensor}\ [43,14] = -\frac{4\,Cot[\theta]\,Csc[\theta]^2}{3\,b^4\,r^2\,(a\,r-t)^3\,(a-\frac{t}{r})^{2/3}}$$

$$\text{KritchmanTensor}\ [43,23] = -\frac{8\,e^{-2\,\phi}\,t\,(a-\frac{t}{r})^{2/3}\,Csc[\theta]^2}{27\,b^2\,r\,(a\,r-t)^5}$$

$$\text{KritchmanTensor}\ [43,31] =$$
$$\left(4\,e^{-2\,(\phi+\varphi)}\left(2\,b^2\,e^{2\,\varphi}\,r^2-b^2\,e^{2\,\phi}\,(-3\,a\,r+t)^2+9\,e^{2\,(\phi+\varphi)}\,r^2\left(a-\frac{t}{r}\right)^{2/3}\right)Csc[\theta]^2\right)\Big/\left(27\,b^4\,r^3\,(-a\,r+t)^4\left(a-\frac{t}{r}\right)^{1/3}\right)$$

$$\text{KritchmanTensor}\ [43,32] = \frac{8\,e^{-2\,\phi}\,t\,(a-\frac{t}{r})^{2/3}\,Csc[\theta]^2}{27\,b^2\,r\,(a\,r-t)^5}$$

$$\text{KritchmanTensor}\ [43,34] =$$
$$-\left(2\,e^{-4\,(\phi+\varphi)}\left(4\,b^2\,e^{2\,\varphi}\,r^2-b^2\,e^{2\,\phi}\,(-3\,a\,r+t)^2+9\,e^{2\,(\phi+\varphi)}\,r^2\left(a-\frac{t}{r}\right)^{2/3}\right)^2\right)\Big/(81\,b^4\,r^4\,(-a\,r+t)^4)+\frac{8\,e^{-2\,\phi}\,(a-\frac{t}{r})^{2/3}\,Csc[\theta]^2}{9\,b^2\,(-a\,r+t)^4}$$

$$\text{KritchmanTensor}\ [43,41] = \frac{4\,Cot[\theta]\,Csc[\theta]^2}{3\,b^4\,r^2\,(a\,r-t)^3\,(a-\frac{t}{r})^{2/3}}$$

$$\text{KritchmanTensor}\ [43,43] =$$
$$\left(2\,e^{-4\,(\phi+\varphi)}\left(4\,b^2\,e^{2\,\varphi}\,r^2-b^2\,e^{2\,\phi}\,(-3\,a\,r+t)^2+9\,e^{2\,(\phi+\varphi)}\,r^2\left(a-\frac{t}{r}\right)^{2/3}\right)^2\right)\Big/(81\,b^4\,r^4\,(-a\,r+t)^4)-\frac{8\,e^{-2\,\phi}\,(a-\frac{t}{r})^{2/3}\,Csc[\theta]^2}{9\,b^2\,(-a\,r+t)^4}$$

Chapter 4

Self-Similar Cosmological Solutions with Scalar Field

4.1 Introduction

Scalar fields have come to play a dominant role in recent years in theoretical models of the universe. This has usually been in the context of inflationary models of the very early universe where the self interaction potential energy density $V(\phi)$ remains undiluted by the cosmological expansion. If the potential is sufficiently flat this can lead to an effective cosmological constant which can drive an accelerated expansion. The detailed nature of the evolution is driven by the specific from of the scalar field's potential energy.

Self-similar homogeneous cosmological solutions are invariant under a global conformal rescaling of the matric. Such models may be expanding, but the physical state at different times differ only by a change in the overall length scale (Wainwright, 1997). In a Friedmann- Robertson-Walker (FRW) cosmological model this is equivalent to a rescaling of the cosmic time,

$$t \rightarrow \Gamma t, \tag{4.1}$$

where Γ is a constant. Self-similarity requires that the evolution of the scale factor $a(t)$ is a power-law, and the Hubble constant $H \equiv \dot{a}/a \propto 1/t$.

We will consider the evolution of spatially flat FRW cosmologies containing a fluid, with density ρ, and pressure P, and a scalar field ϕ with self-interaction potential $V(\phi)$. In general relativity, the Friedmann constant equation requires

$$H^2 = \frac{8\pi G_N}{3}\left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{4.2}$$

where G_N is Newton's constant. Equation (4.1) then requires that the matter content is invariant under a rescaling $\rho \rightarrow \Gamma^{-2}\rho$. Requiring $P \rightarrow \Gamma^{-2}P$ for all ρ then leads to a barotropic equation of state $P = (\gamma - 1)\rho$ where γ is a constant. Thus the familiar radiation ($\gamma = 4/3$) or pressureless matter ($\gamma = 1$) dominated FRW models can be described as self-similar solution with

$$a \propto t^{2/(3r)}. \tag{4.3}$$

A scalar field has kinetic energy $\dot{\phi}^2/2$ which only allows a constant shift of the scalar field, $\phi \rightarrow \phi + \phi_\Gamma$, if we are to require $\dot{\phi}^2 \rightarrow \Gamma^{-2}\dot{\phi}^2$. In order to obtain a self-similar solution we require in addition

$$V(\phi) \to V(\phi + \phi_\Gamma) = \Gamma^{-2} V(\phi). \tag{4.4}$$

which is only compatible with a non-interacting field, $V(\phi) = 0$, or an exponential potential

$$V(\phi) = V_0 exp(-\lambda \kappa \phi), \tag{4.5}$$

where $\kappa^2 = 8\pi G_N$, λ is a dimensionless constant and we have $\kappa\phi_\Gamma = (2/\lambda) ln\Gamma$. FRW solutions for scalar fields with exponential potentials where the kinetic energy and potential energy of the field remain proportional were proposed as a model for power-law inflation in the early universe by Luchinn and Matarrese (Luchinn and Matarrese, 1993), and are the late-time attractor solutions (in the absence of other matter) for $\lambda^2 < 6$. More recently attention has focused on the possible late-time evolution of scalar fields in FRW cosmologies containing matter. Self-similar solutions are known for scalar fields with exponential potentials whose energy density scales with that of a barotropic fluid yielding the same time dependence given in Eq.(4.3) for a barotropic fluid dominated solution (Wetterich, 1988). These scaling solutions are the unique late-time attractors for sufficiently steep potentials $\lambda^2 > 3_\gamma$ (Copeland, 1988). Such a scalar fields is so successful at scaling with the barotropic matter that the scalar field never comes to dominate the cosmological dynamics.

In the next section we show that self-similar cosmological solutions are possible for scalar fields with simple power-law potentials if the scalar field has motion reduce to an autonomous phase-plane whose fixed points correspond to self-similar solutions.

4.2 Brans-Dicke-Type Cosmology with Power-Law Potential

It is possible to obtain self-similar solutions for scalar fields with arbitrary power-law potentials, $V(\phi) = V_0(\kappa\phi)^{2n}$, if we go beyond Einstein's theory of general relativity and allow the field to be non-minimally coupled to the spacetime curvature. In Brans-Dicke gravity (Brans and Dicke, 1961) with dimensionless parameter ω, Newton's constant is replaced by a dynamical field $G = \omega/(2\pi\phi^2)$ and the generalised Friedmann equation requires

$$H^2 = \frac{8\pi}{3}\frac{\omega}{2\pi\phi^2}\left(\rho + \frac{1}{2}\dot{\phi}^2 + \frac{3}{2\omega}H\phi\dot{\phi}\right). \qquad (4.6)$$

In the original Brans-Dicke theory where $V(\phi) = 0$, eqn(4.1) is compatible with a global rescaling of the barrotropic fluid density $\rho \to \Gamma_\rho \rho$ and $\phi \to \Gamma_\phi \phi$for all $\Gamma_\rho/\Gamma_\phi^2 = \Gamma^{-2}$. The self-similar solution for a barotropic fluid in Brans-Dicke gravity was given by (Nariai, 1968). In the presence of a power-low potential, $V(\phi) = V_0(\kappa\phi)^{2n}$, we require in addition that $V(\phi)/\phi^2 \to \Gamma^{-2}[V(\phi)/\phi^2]$ which specifies $\phi \to \Gamma^{-1/(n-1)}\phi$ and $\rho \to \Gamma^{-2n/(n-1)}\rho$. Thus self-similar solutions can exist for "quintessence"-type power-law scalar field potentials (Caldwell et al, 1998), so long as this scalar field has Brans-Dicke type coupling to the spacetime curvature.

scalar field has Brans-Dicke type coupling to the spacetime curvature. Motivated by the preceding qualitative discussion of self-similar solutions we examine the cosmological evolution of one of the simplest forms of non-minimal coupling proposed by (Brans-Dicke, 1961) which is described by a single dimensionless parameter ω. In addition to the original Brans-Dicke lagrangian (Brans-Dicke, 1961), we introduce self interaction potential $V(\phi)$ (Barro, 1990). The action is

$$S = \int \sqrt{-g} d^4x [\pm \frac{1}{8\omega} \phi^2 R \mp \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi)] + \int \sqrt{-g} d^4x \mathcal{L}_{matter} \tag{4.7}$$

where upper/lower signs should be chosen to ensure $\pm\omega > 0$and hence a positive gravitational coupling. The self-interaction potential is taken to be a power-law,

$$V(\phi) = V_0 (\kappa\phi)^{2n}. \tag{4.8}$$

This action can be re-expressed as a theory of interacting matter fields in general relativity with fixed Newton's constant G_N (Dicke, 1962) if we define quantities in the conformally related Einstein frame with respect to the rescaled metric

$$\tilde{g}_{\mu\nu} = \left| \frac{\kappa^2 \phi^2}{4\omega} \right| g_{\mu\nu}. \tag{4.9}$$

The scalar field is now minimally coupled to metric $\tilde{g}_{\mu\nu}$, but non-minimally coupled to the other matter fields. We will re-express the scalar field ϕ in terms of a field ψ which has a canonical kinetic term in the Einstein frame, which requires that

$$\kappa\phi = exp\left(\frac{\kappa\psi}{\sqrt{2(3+2\omega)}}\right). \tag{4.10}$$

The scalar field will have a non-negative energy density in the Einstein frame so long as $\omega > -3/2$. Thus we consider ω with any value $> -3/2$, which includes, for example, the coupling of the string-theory dilaton where $\omega = -1$ (Schwarz, 1988).

The effective self-interaction potential for canonical field ψ in the Einstein frame is of exponential form and given by Coley et al (Coley et al, 1999)

$$\tilde{V}(\psi) = \left(\frac{4\omega}{\kappa^2\phi^2}\right)^2 V(\phi) - \tilde{V}_0 exp(-\lambda\kappa\psi), \tag{4.11}$$

where $\tilde{V}_0 \equiv (4\omega)^2 V_0$ and

$$\lambda \equiv \sqrt{\frac{2}{3+2\omega}}(2-n). \tag{4.12}$$

4.3 Autonomous Phase-Plane

Using the Hubble rate, $\tilde{H}$, fluid density, $\tilde{\rho}$, and scalar field density, $\tilde{\rho}_\psi = (1/2)(d\psi/d\tilde{t})^2 + \tilde{V}(\psi)$, defined in the Einstein frame, the equations are

$$\frac{d\tilde{H}}{d\tilde{t}} = -\frac{\kappa^2}{2}\left(\tilde{\rho} + \tilde{P} + \left(\frac{d\psi}{d\tilde{t}}\right)^2\right), \tag{4.13}$$

$$\frac{d\tilde{\rho}}{d\tilde{t}} + 3\tilde{H}(\tilde{\rho} + \tilde{P}) = -\frac{dQ}{d\tilde{t}}, \tag{4.14}$$

$$\frac{d\psi}{d\tilde{t}}\left(\frac{d^2\psi}{d\tilde{t}^2} + \frac{d\tilde{V}}{d\psi} + 3\tilde{H}\frac{d\psi}{d\tilde{t}}\right) = \frac{dQ}{d\tilde{t}}, \tag{4.15}$$

subject to the Friedmann constraint,

$$\tilde{H}^2 = \frac{\kappa^2}{3}\left(\tilde{\rho} + \frac{1}{2}\left(\frac{d\psi}{d\tilde{t}}\right)^2 + \tilde{V}(\psi)\right). \tag{4.16}$$

The non-minimal coupling of the scalar field in the Brans-Dicke frame leads to an explicit energy transfer in the Einstein frame, between the scalar field and the fluid,

$$\frac{dQ}{d\tilde{t}} \equiv \kappa \frac{d^2\psi}{d\tilde{t}} \sqrt{\frac{\tilde{\rho} - 3\tilde{P}}{3 + 2\omega}}. \tag{4.17}$$

The same system of equations was considered, starting from a different motivation, by Wetterich (Wetterich, 1995), and more reently by Amendola (Amendola, 1999). In the case $dQ/d\tilde{t} = 0$, these equations reduce to previous studied of minimally coupled scalar fields with exponential potentials (Copeland et al, 1998).

We define,

$$x \equiv \frac{\kappa}{\sqrt{6}\tilde{H}}\left(\frac{d\psi}{d\tilde{t}}\right), \qquad y \equiv \frac{\kappa\sqrt{\tilde{V}}}{\sqrt{3}\tilde{H}}. \tag{4.18}$$

The evolution equations for a barotropic fluid where $\tilde{P} = (\gamma - 1)\tilde{\rho}$ can then be written as

$$x' = -3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x[2x^2 + \gamma(1 - x^2 - y^2)] + W(1 - x^2 - y^2), \tag{4.19}$$

$$y' = -\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y[2x^2 + \gamma(1 - x^2 - y^2)], \tag{4.20}$$

straint equation,

$$\frac{\kappa^2\tilde{\rho}}{3\tilde{H}^2} + x^2 + y^2 = 1. \qquad (4.21)$$

We parameterise the energy transfer in terms of the dimensionless quantity W, where

$$W \equiv \sqrt{\frac{3}{3+2\omega}}\left(\frac{4-3\gamma}{2}\right), \qquad (4.22)$$

eqns (4.19) and (4.20) define an two-dimensional autonomous phase-plane whenever W can be written as function of x and y. The Brans-Dicke-type theory defined in eqn(4.7) naturally leads to an interaction with constant W, although the analysis could be extended to more general $W(x, y)$ (Billyard, 1999).

The constraint eqn (4.21) restricts physical solutions with non-negative fluid density to $0 \leq x^2 + y^2 \leq 1$ and so the evolution is completely described by trajectories within the unit disc. In the following discussion we will only consider expanding cosmologies corresponding to the upper-half disc, $y \geq 0$.

4.4 Self-Similar Solutions

Fixed points of the system, (x_i, y_i) where $x' = 0$ and $y' = 0$,correspond to power-law solutions for the scale factor and logarithmic evolution of the scalar field with respect to the cosmic time in the Einstein frame:

$$\tilde{a} \propto \tilde{t}^{\tilde{p}}, \qquad \kappa\psi \propto \tilde{q} ln\tilde{t}, \qquad (4.23)$$

where the constants $\tilde{p}$ and $\tilde{q}$ are given by,

$$\tilde{p} = \frac{2}{3\gamma(1 - x_i^2 - y_i^2) + 6x_i^2},$$

$$\tilde{q} = \frac{2\sqrt{6}x_i}{3\gamma(1 - x_i^2 - y_i^2) + 6x_i^2}. \tag{4.24}$$

These are self-similar solutions where the dimensionless density parameter for the barotropic fluid in the Einstein frame, given from the constant eqn(4.21)

$$\tilde{\Omega} \equiv \frac{\kappa^2 \tilde{\rho}}{3\tilde{H}^2} = 1 - x_i^2 - y_i^2, \tag{4.25}$$

is a constant and therefore so is the dimensionless density of the scalar field

$$\tilde{\Omega}_\psi = x_i^2 + y_i^2. \tag{4.26}$$

The scalar field has an effective barotropic index given by,

$$\tilde{\gamma}_\psi = \frac{\tilde{P}_\psi + \tilde{\rho}_\psi}{\tilde{\rho}_\psi} = \frac{2x^2}{x^2 + y^2}. \tag{4.27}$$

The existence and stability of critical points, already defined in the Einstein frame, remain unchanged under a conformal transformation back to the Brans-Dicke frame. The evolutionary behaviour for the scale factor and scalar field is however modified. If we denote the evolutionary behaviour of the scale factor a and scalar field ϕ in the Brans-Dicke frame by,

$$a \sim t^p, \qquad \phi^2 \sim t^q \tag{4.28}$$

exponents can be related to those in the Einstein frame by,

$$p = \tilde{p} + \frac{(\tilde{p} - 1)\tilde{q}}{\sqrt{2(3 + 2\omega)} - \tilde{q}},$$

$$q = \frac{2\tilde{q}}{\sqrt{2(3+2\omega)} - \tilde{q}}. \tag{4.29}$$

Depending on the values of the parameters λ, γ and W we can have up to five fixed points in the Einstein frame.The nature and stability of each point in the phase-plane is briefly below.

4.4.1 2- Way, Matter- Kinetic Scaling Solution

$$x_1 = \frac{2W}{3(2-\gamma)}, \qquad y_1 = 0. \tag{4.30}$$

This solution lies on the x-axis where the scalar field potential is negligible, and the scalar field's density in the Einstein frame is dominated by its kinetic energy, leading to a stiff equation of state for the scalar field, $\gamma_\psi \bar{=} 2$. We have a power-law solution of the form given in eqn(4.23) with

$$\tilde{p}_1 = \frac{6(2-\gamma)}{9\gamma(2-\gamma)+4W^2},$$

$$\tilde{q}_1 = \frac{4\sqrt{6}W}{9\gamma(2-\gamma)+4W^2}. \tag{4.31}$$

In the Brans-Dicke frame these are the power law solutions for a barotropic fluid in Brans-Dicke gravity as first given by (Nariai, 1968), where the power law exponents of the scale factor and scalar field are given by

$$p_1 = \frac{2\omega(2-\gamma)+2}{3\omega\gamma(2-\gamma)+4}$$

$$q_1 = \frac{2(4-3r)}{3\omega\gamma(2-\gamma)+4}. \tag{4.32}$$

4.4.2 Kinetic Dominated Solutions

$$x_{2,3} = \pm 1, \qquad y_{2,3} = 0 \tag{4.33}$$

These solutions correspond to negligible fluid density and negligible scalar field potential so that the Friedmann constraint equation is dominated by the kinetic energy of the scalar field with a stiff equation of state $\tilde{\gamma}_\psi = 2$. We have power-law solutions of the form given in Eq.() with

$$\tilde{p}_{2,3} = 1/3, \qquad \tilde{q}_{2,3} = \pm\sqrt{2/3}. \tag{4.34}$$

In the Brans-Dicke frame we recover the vacuum solutions of Brans-Dicke gravity found by (O'Hanlon , 1972) with power-law exponents given by

$$p_{2,3} = \frac{\sqrt{6} \mp \sqrt{2(3+2\omega)}}{\sqrt{6} \mp 3\sqrt{2(3+2\omega)}},$$
$$q_{2,3} = \frac{\pm 2\sqrt{6}}{3\sqrt{2(3+2\omega)} \mp \sqrt{6}}. \tag{4.35}$$

4.4.3 Scalar Field Dominated Solutions

$$x_4 = \lambda/\sqrt{6}, \qquad y_4 = (1 - \lambda^2/6)^{1/2} \tag{4.36}$$

Here the fluid density is negligible, but neither the kinetic energy, nor the potential dominates the energy density of the scalar field in the Einstein frame. The scalar field has an effective equation of state $\tilde{\gamma}_\psi = \lambda^2/3$, and for

$\lambda^2 < 2$ this solution correspond to power-law inflation (Lucchin, 1985). For $\lambda \neq 0$ we have power-law solutions of the form given in eqn(2.18) with

$$\tilde{p}_4 = \frac{2}{\lambda^2}, \qquad \tilde{q}_4 = \frac{2}{\lambda}. \tag{4.37}$$

In the Brans-Dicke frame the power-law exponents are given (for $n \neq 1$ and $n \neq 2$) by (Barrow, 1990)

$$\begin{aligned} p_4 &= \frac{2\omega + n + 1}{(2-n)(1-n)}, \\ q_4 &= \frac{2}{1-n} \end{aligned} \tag{4.38}$$

For $n = 0$ the potential acts like a false vacuum energy density in the Brans-Dicke frame and this corresponds to extended inflation solutions for $\omega > 1/2$. For $n = 2$ it is the potential in the Einstein frame that remains constant, leading to de Sitter expansion. The case $n = 1$ was studied by (Kolitch, 1996).

4.4.4 3-Way, Matter-Kinetic-Potential Scaling Solutions

$$\begin{aligned} x_5 &= \frac{3\gamma}{\sqrt{6}\lambda - 2W}, \\ y_5^2 &= \left[\frac{9\gamma(2-\gamma) - 2W(\sqrt{6}\lambda - 2W)}{(\sqrt{6}\lambda - 2W)^2}\right]. \end{aligned} \tag{4.39}$$

Here neither the fluid nor the scalar field dominates the evolution, and we have self-similar solution where both the potential and kinetic energy density of the scalar field remains proportional to that of the barotropic matter. The effective equation of state for the scalar field is given by,

$$\tilde{\gamma}_{\psi 5} = \gamma \left(\frac{6\gamma}{6\gamma - W(\sqrt{6}\lambda - 2W)} \right) . \tag{4.40}$$

We have power-law solutions of the form given in eqn(4.18) with (Amendola, 1999)

$$\tilde{p}_5 = \frac{2}{3\gamma} \left(\frac{\sqrt{6}\lambda - 2W}{\sqrt{6}\lambda} \right) , \qquad \tilde{q}_5 = \frac{2}{\lambda} . \tag{4.41}$$

In the Brans-Dicke frame this corresponds to power-law evolution with exponts given by,

$$p_5 = \frac{2}{3\gamma} \left(\frac{n}{n-1} \right) , \qquad q_5 = \frac{2}{1-n} . \tag{4.42}$$

As far as we are aware, this solution has not been discussed before in the context of Brans-Dicke gravity. It is interesting to note that the cosmological evolution in the Brans-Dicke frame is independent of the Brans-Dicke parameter ω, although it does determine the existence of this 3-way scaling solution.

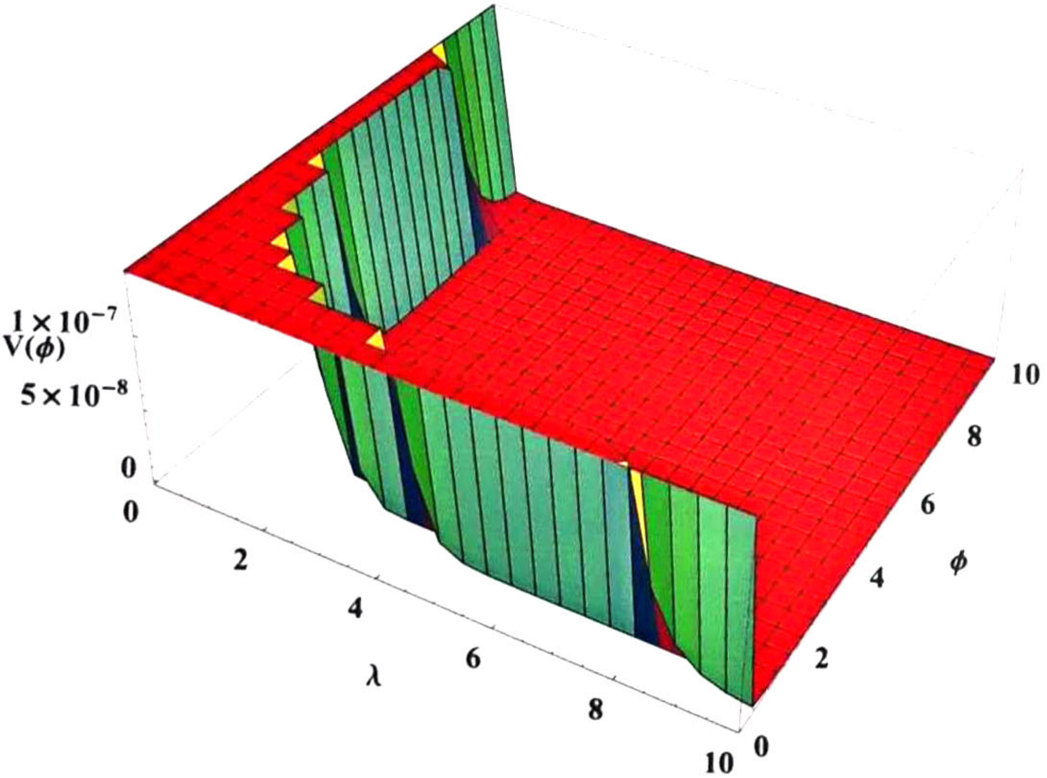

Figure 4.1: The profile of the self-similar self-interaction potential of scalar field $V(\phi)$ with scalar field ϕ and dimensionless constant λ

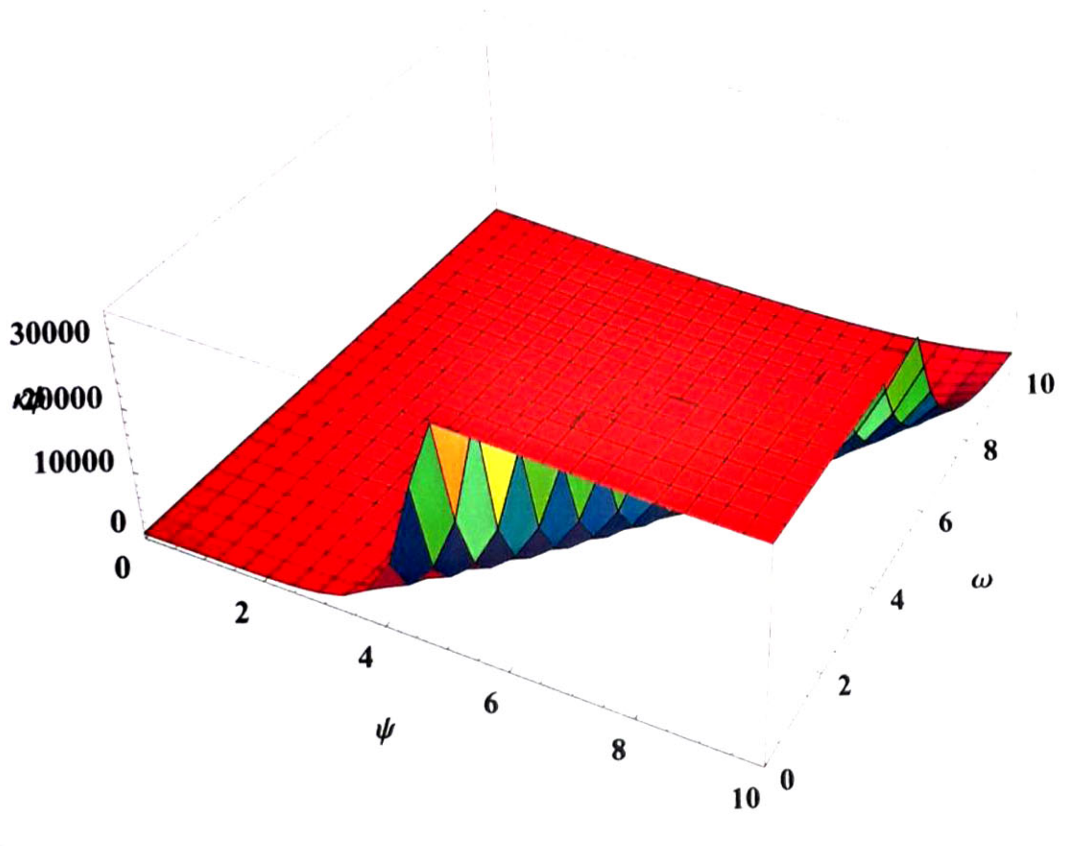

Figure 4.2: The profile of the scalar field $\kappa\phi$
with transformed scalar field ψ and dimensionless parameter ω

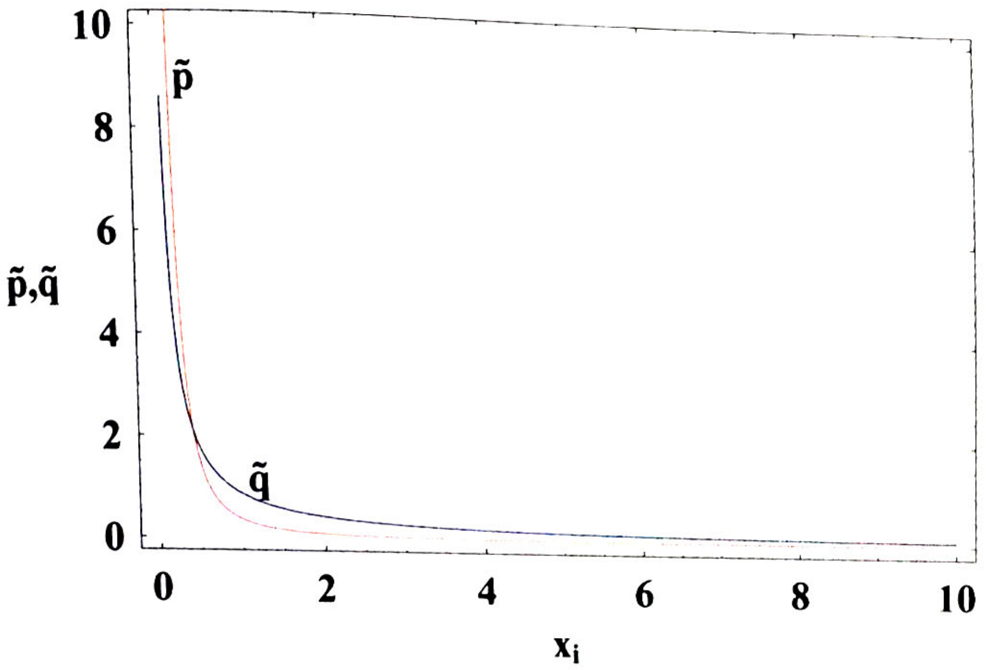

Figure 4.3: The comparision of the dimensionless constants $\tilde{p}$ and $\tilde{q}$ of the scalar field with varying the function of x_i

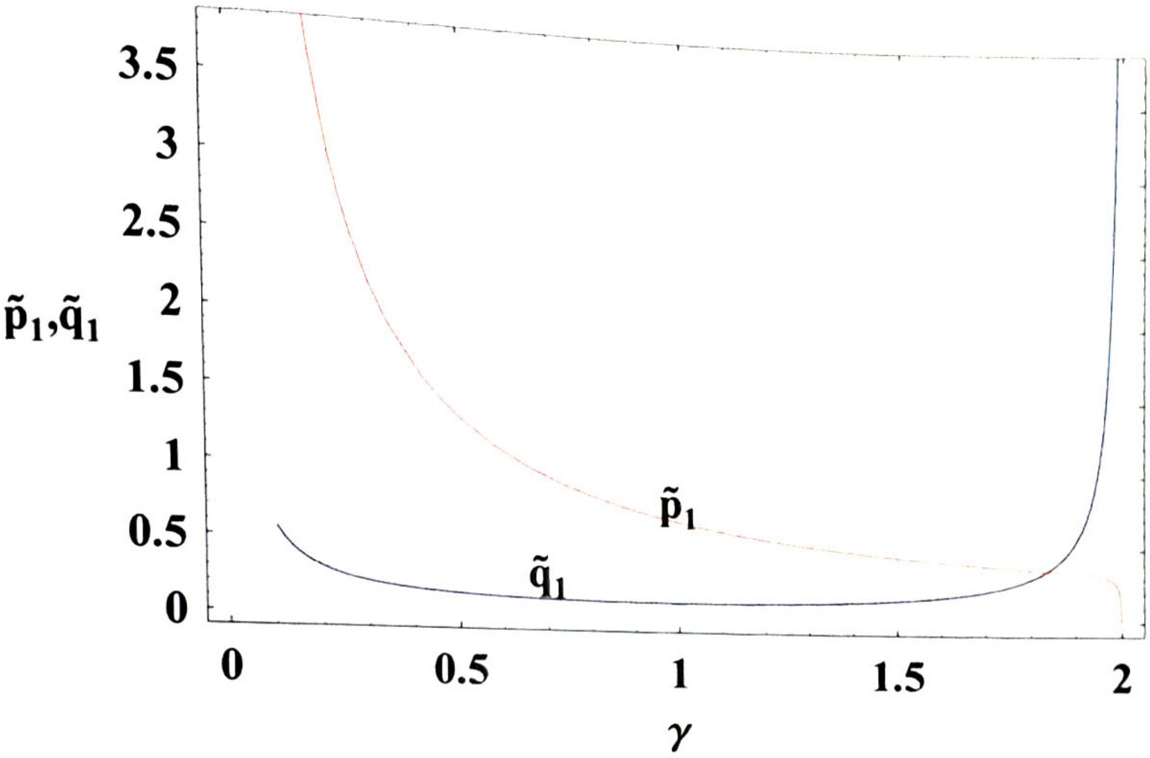

Figure 4.4: The comparision of the dimensionless constants $\tilde{p}_1$ and $\tilde{q}_1$ of the scalar field with varying the function of γ

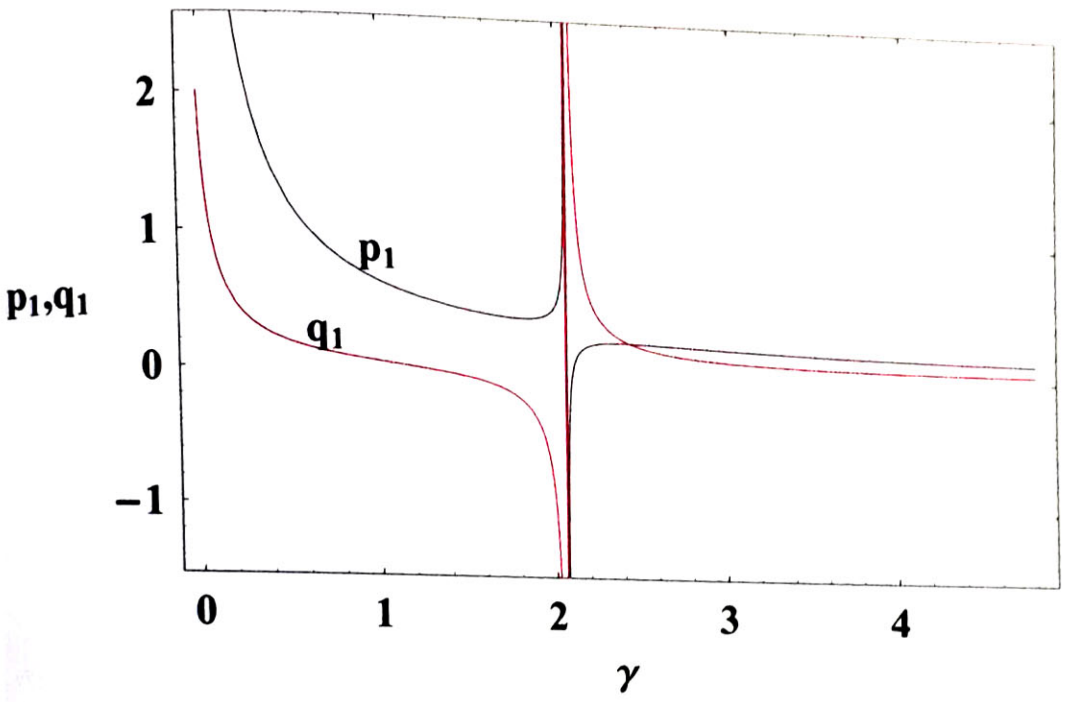

Figure 4.5: The comparision of the dimensionless constants p_1 and q_1 of the scalar field with varying the function of γ

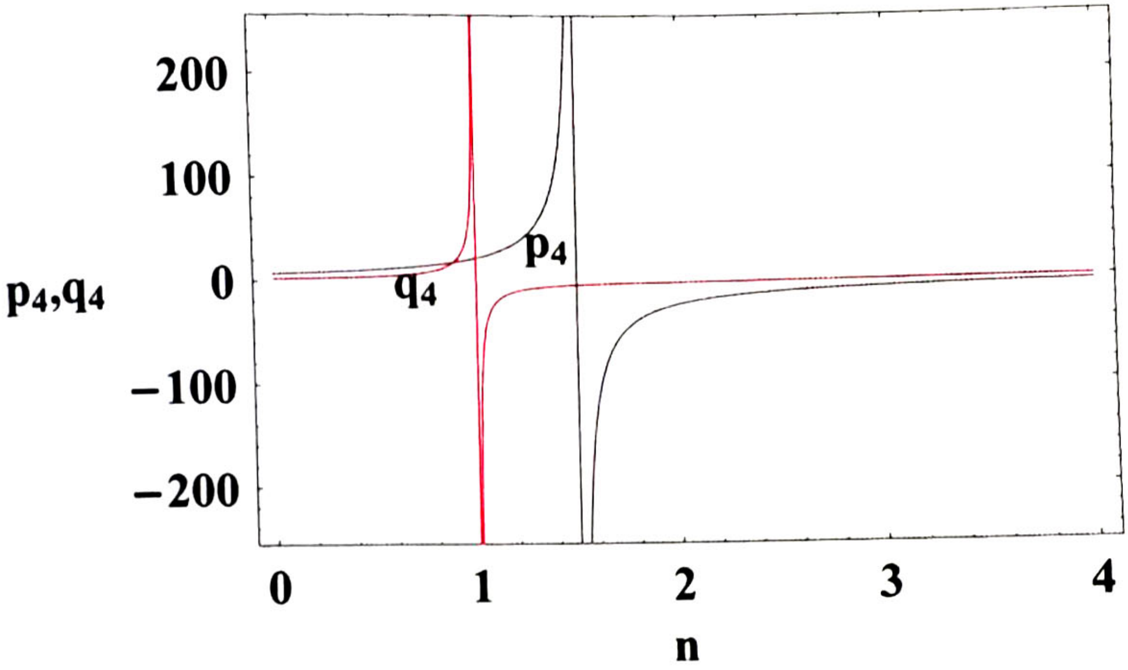

Figure 4.6: The comparision of the dimensionless constants p_4 and q_4 of the scalar field with varying the function of n

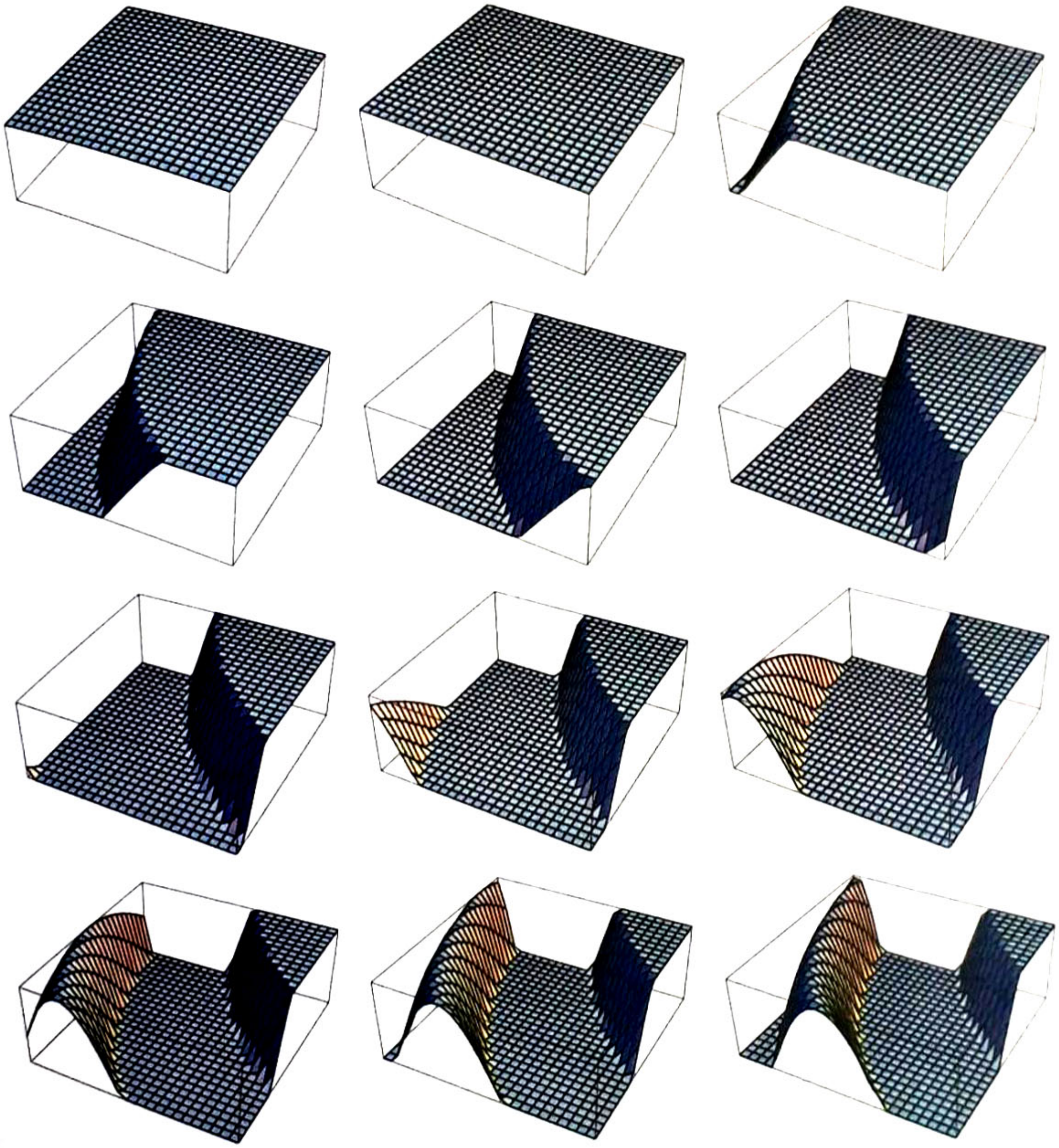

Figure 4.7: The animation plot of an exponantial potential $V(\phi)$

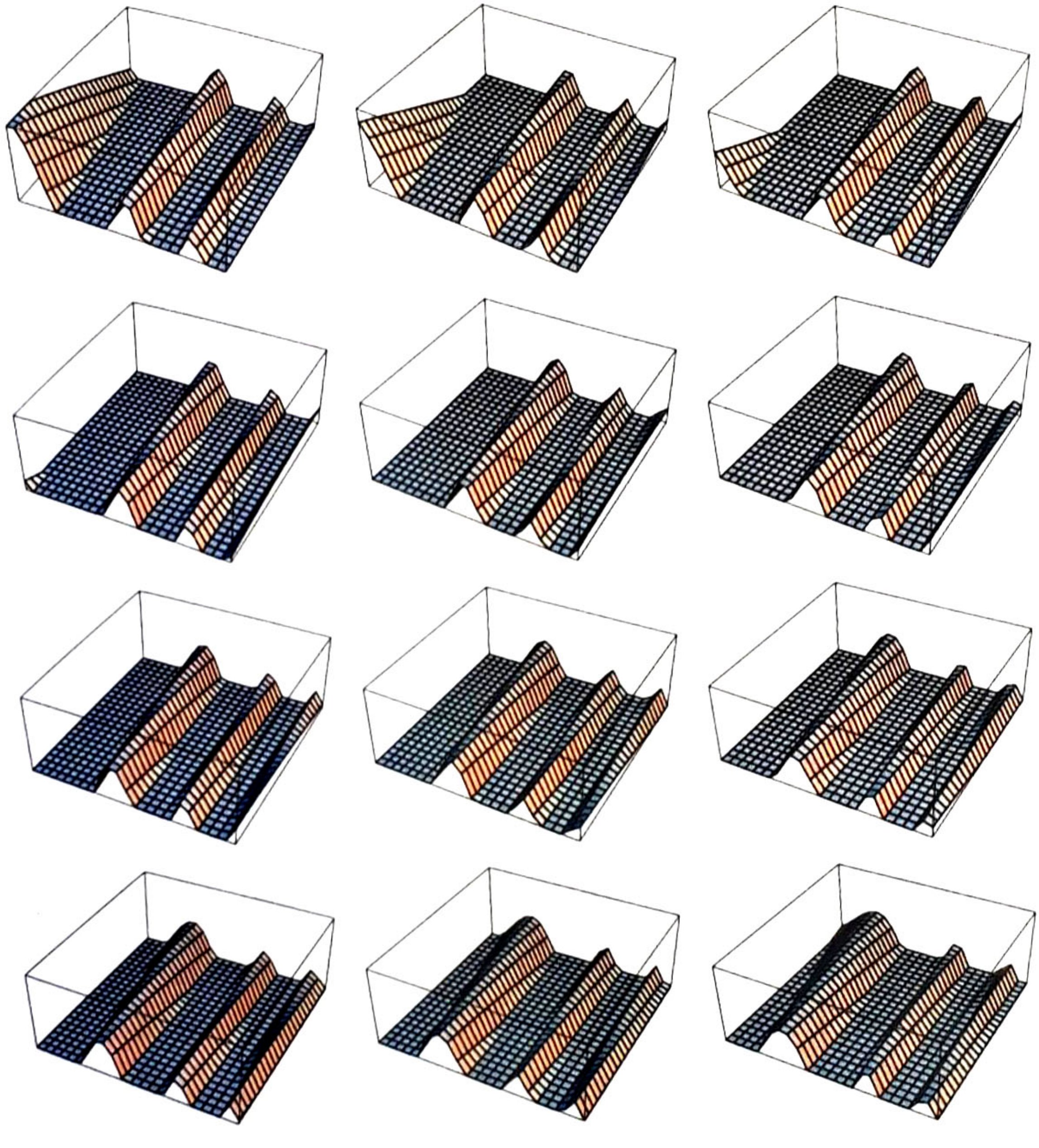

Figure 4.8: The animation plot of the variable Hubble parameter H

Chapter 5

Results and Concluding Remarks

In this thesis some fundamental notions and basics of self-similar spacetime are given and attempts are made to derive the path of photon using simple tractable methods. The Kretschmann scalar is also calculated for spherical symmetric spacetimes metric. It is possible to examine the curvature strength of naked singularity. It has been found that the points of unboundedness occur at (ar, r), i.e, self-similar points. At this point it can be noted that the case of the naked singularity arising due to gravitational collapse of self-similar null dust. The origin of the coordinate there is a naked singularity which is a node and an entire family of non-spacelike geodesics escape, exposing the singularity to a distant observer for an infinite time. The possible path of photon has been derived explicitly and visualization of this possible path is done.

The introduction of self-similarity into Newtonian gravity is straightforward because it is the theory with absolute space time. Since Newtonian gravity has only one dimensional constant, i.e, the gravitational constant, we can incorporate it into a polytropic gas as well as isothermal gas to transform them into the frame work of similarity. Self-similar solutions are derived and visualizations of some results are done with the help of Mathematica.

The introduction of self-similarity to general relativity is not so straightforward since there is no prefered coordinate system in this theory. The covariant definition of complete similarity is homothety, and it is impossible to incorporate many physically interesting quantities, such as a polytropic equation of state, into the framework of homothety. It has been assumed that one of the most natural definitions of incomplete similarity in the fluid system in general relativity is kinematic self-similarity. Some self-similar Einstein's field solutions are obtained using tensorpack.m and some interesting iterations carried out with the help of Mathematica.

Self-similarity cosmological solutions with scalar field potential have been treated in chapter IV. It is advantageous to use the fact that self-similar cosmological solutions are invariant under global conformal rescaling of the metric and they would differ only by a change in overall length scale. Rescaling of the physical quantities, including scalar field and scalar potential, we are able to re-express the scalar field ϕ in terms of a field ψ. Autonomous phase plane, self-similar and scaling exact solutions with scalar field are given and some animation plots are visualized using Mathematica.

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